A Weighted L_2 Based Method for the Design of Arbitrary One Dimensional FIR Digital Filters

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Abstract

In FIR filter design problems, filter specifications do not constrain in any way the ideal frequency response inside the transition regions. Most existing L_2 based filter design techniques utilize this flexibility in order to improve their performance. In this paper we propose a new general method for the weighted L_2 based design of arbitrary FIR filters. In particular we propose a well defined optimization criterion that depends on the selection of the desired response inside the transition regions. By optimizing our criterion we obtain desired responses that produce weighted mean square error optimum filters with extremely good characteristics. The proposed method is computationally simple since it requires the solution of a linear system of equations.

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1 Introduction

The design of one-dimensional (1-D) digital filters, although it is an old problem with significant existing literature, has been of growing interest over the last decade. This is because digital filters are widely used in a variety of signal processing applications such as speech and image processing, communications, seismology, radar, sonar and medical signal processing.

The best known class of 1-D FIR filters is the class of linear phase filters. Their popularity stems mainly from the fact that corresponding design methods involve only real functions which also allows for the successful employment of the L_∞ criterion, the most suitable one, for the filter design problem. Linear phase filters are known however to introduce significant delays when their lengths are large. In applications where long delays are unacceptable, it is clear that there is a need of alternative filters. Furthermore there are problems which are by nature nonlinear-phase such as, piecewise constant group delay FIR filters, FIR equalizers, beamformers, seismic migration filters etc. These problems require a filter design methodology that is significantly different from the conventional used in linear phase. Moreover one can no longer be limited to real coefficient filters and needs to take into account general complex coefficient filters.

What constitutes the complex function design problem challenging, from a methodology point of view, is the lack of efficient L_∞ techniques as compared to linear phase where the Remez Exchange Algorithm [17] is dominant. This is largely due to the non-existence of a suitable counterpart, in the complex case, to the Alternation Theorem [13] that can serve as a base for developing computationally efficient algorithms. Consequently all L_∞ techniques rely on sophisticated and computationally intense optimization machinery.

Existing L_∞ techniques can be broadly classified into four categories. The first includes methods that replace the complex L_∞ approximation problem with two real ones and therefore their solution is suboptimal [8], [19]. The second includes methods that solve the L_∞ problem using linear programming [8], [25], [2], [3]. The third category involves methods based on Lawson's algorithm (also known as iterated reweighted least squares), [7], [9], [26]. Finally, the last category includes methods that enforce an equiripple structure on the solution [20], [24] which however does not guarantee L_∞ optimality [27].

It should be noted that all the above techniques, due to their high computational complexity, fail to yield meaningful results even for moderate size filters. It is therefore clear that the computationally simpler and therefore more robust L_2 based methods become more tractable in the complex case than they are in linear phase problems, where the Remez algorithm provides the required solution quite efficiently.

The L_2 criterion, in the case of constant weight, results in the well known Fourier series coefficients which, in most cases, can be easily obtained analytically. The poor performance of this classical method (due to the Gibbs' phenomenon) can be improved by introducing transitions regions between the passbands and stopbands of the filter, an idea also used in the linear phase case [4], [5], [6], [16, p. 70], [28]. Therefore existing complex filter design techniques [6], [12], [14], [16, p. 130] mainly become generalizations to their real linear phase counterparts. In particular, two of the above methods, the "don't care" [12], [14], [16, p. 130] and the eigenfilter [15], [18], have comparable performance [12] and their computational complexity is also moderate. Specifically, the first requires the solution of a linear system of equations, whereas the second the computation of the eigenvector corresponding to the smallest eigenvalue of a symmetric matrix (problem that can be treated using SVD or the iterative power method). Both methods succeed in reducing the Gibbs' phenomenon. Furthermore, the above mentioned techniques can be easily modified to include a variable weighting function. This seems not to be an easy task for other well known with very low complexity techniques [4], [5] which, in the case of constant weight, are proved in [11] to be L_2 optimal.

In [21], an L_2 based method suitable for the unweighted design of the zero phase FIR digital filters was introduced. The basic characteristic of this method is that it is capable of optimally defining the unknown part of the ideal frequency response inside the transition regions and drastically reduce the Gibbs' phenomenon. The computational complexity of the method is low since it requires the solution of a set of linear equations. Finally the method outperforms the most popular non L_∞ design techniques while it compares very favorably with the actual L_∞ optimum solution. In this work we intend to extend this idea to the complex filter design case and also include variable weighting function. The method we developed, as we are going to see in the Design Examples section, even outperforms the "Complex Remez" algorithm of [13] that has been included in Matlab, for the complex filter design problem. In fact this better performance is in the sense of designing filters with smaller maximum deviation but also with a significantly smaller computational complexity.

At this point we must note that the method we are going to present is not applicable to Constrained or to Peak Constrained Least Squares filter design problems. More specifically, the proposed method can not be applied to cases in which we are interested in using different measures in different frequency bands [1], [7], or to cases in which we like to take into account peak error constraints during the minimization of the weighted least square criterion [1], [22], [23]. In such filter design problems other well known filter design techniques [1], [7], [22], [23] (and the references there in) are better suited and should be used.

The paper is organized as follows, Section 2 contains a brief description of the weighted and unweighted Fourier approximation problem. Section 3 contains our main result, namely the proposed filter design method. In Section 4 we apply our method in a number of 1-D filter design examples and make comparisons with other existing filter design techniques. Finally Section 5 contains our conclusions.

2 Background Material

It is well known that the filter design problem, when considered in the frequency domain, is equivalent to a function approximation problem. Since frequency responses are periodic functions with period 2π we can limit ourselves to the frequency interval $[-\pi, \pi]$. Suppose that the complex function $d(\omega)$, defined on the interval $[-\pi, \pi]$, denotes the desired frequency response. We like to approximate $d(\omega)$ using linear combinations of the complex exponentials $e^{-jn\omega},\ n = N_1, \ldots, N_2$, with the coefficients of the combination constituting the filter coefficients.

In this work we consider only the case $N_2 - N_1$ being an even integer since the odd case can be treated similarly. Without loss of generality we can assume that $-N_1 = N_2 = N$. This is so because it can be proved [13] that, approximating $d(\omega)$ with linear combinations of $e^{-jn\omega},\ n = N_1, \ldots, N_2$, is equivalent to approximating the complex function $d(\omega)e^{j0.5(N_1+N_2)\omega}$ with linear combinations of $e^{-jn\omega},\ n = -(N_2 - N_1)/2, \ldots, (N_2 - N_1)/2$.

Although we shall employ an L_2 based design approach, we must stress that in conventional filter design, the L_∞ norm is considered as the most appropriate criterion. In fact, even if one is using alternative criteria (as L_2), the implicit goal is to approximate the L_∞ optimum filter as close as possible. Therefore, from now on, when we refer to the "performance" of a filter design method we will mean the remaining L_∞, i.e. maximum approximation error.

Let now $f(\omega)$, $g(\omega)$ be two functions defined on $[-\pi, \pi]$, we can then define their usual inner product as

$$< f, g >= \int_{-\pi}^{\pi} f(\omega) g^*(\omega) d\omega, \tag{1}$$

where superscript "*" denotes complex conjugate. Similarly if $\mathbf{f}(\omega) = [f_1(\omega) \ \cdots \ f_k(\omega)]^t$ and $\mathbf{g}(\omega) = [g_1(\omega) \ \cdots \ g_m(\omega)]^t$ are two vector functions then $< \mathbf{f}, \mathbf{g} >$ denotes a matrix of dimensions $k \times m$ defined as

$$< \mathbf{f}, \mathbf{g} >= \int_{-\pi}^{\pi} \mathbf{f}(\omega) \mathbf{g}^H(\omega) d\omega, \tag{2}$$

where the superscript "^{H}" denotes conjugate transpose (hermitian). Finally with the help of the inner product we can define the norm of a scalar function $f(\omega)$ as $\|f\| = \sqrt{< f, f >}$.

Consider now the following vector function of ω

$$\mathbf{e}(\omega) = [e^{-jN\omega} \ e^{-j(N-1)\omega} \cdots e^{j(N-1)\omega} \ e^{jN\omega}]^t, \tag{3}$$

where superscript "^{t}" denotes transpose, if $\mathbf{h} = [h_{-N} \ h_{-N+1} \ \cdots \ h_{N-1} \ h_N]^t$ is the vector of filter coefficients then the filter frequency response can be written as

$$h(\omega) = \mathbf{e}^H(\omega)\mathbf{h}. \tag{4}$$

Our goal is to select the coefficient vector $\mathbf{h}$ so as the corresponding function $h(\omega)$ approximates a desired response $d(\omega)$ optimally. It is clear that the final form of the optimum filter strongly depends on the optimality criterion we employ. If the desired response $d(\omega)$ is completely known over the whole interval $[-\pi, \pi]$ and we select $\mathbf{h}$ to minimize the mean square error $\min_{\mathbf{h}} \|d - h\|^2$ then, as it is well known, the optimum coefficients reduce to the Fourier series

$$\mathbf{h}_o = \frac{1}{2\pi} < \mathbf{e}, d^* > . \tag{5}$$

In filter design problems we very often use a weighting function $w(\omega) > 0$ in order to emphasize certain frequency regions. If this function is again known over the whole frequency interval then we can determine the filter coefficients by employing a weighted least squares criterion of the form $\min_{\mathbf{h}} \|w(d - h)\|^2$. The corresponding optimum filter is then given [14] by

$$\mathbf{h}_o =< w\mathbf{e}, w\mathbf{e} >^{-1}< w\mathbf{e}, wd^* > . \tag{6}$$

and involves the solution of a linear system of equations. As was mentioned before, least squares solutions are known to suffer from the undesirable Gibbs' phenomenon which is particularly pronounced at frequencies where the desired response exhibits discontinuities.

When the desired response is not defined over the whole frequency interval $[-\pi, \pi]$, in other words, when transition regions are introduced in order to avoid discontinuities, then the mean square error criterion can be modified to account for this partial knowledge. Specifically we can perform the following minimization $\min_{\mathbf{h}} \|w(d-h)\mathbb{1}_{\mathcal{U}}\|^2$ where $\mathcal{U}$ is the union of all frequency regions where the desired response is known, that is, the union of the passbands and stopbands and $\mathbb{1}_{\mathcal{U}}(\omega)$ denotes the indicator function of the set $\mathcal{U}$. In other words the mean square criterion takes into account only the frequency regions where the desired response is known. The optimum solution has the following form

$$\mathbf{h}_{dc} = < w\mathbf{e}\mathbb{1}_{\mathcal{U}}, w\mathbf{e}\mathbb{1}_{\mathcal{U}} >^{-1} < w\mathbf{e}\mathbb{1}_{\mathcal{U}}, wd^*\mathbb{1}_{\mathcal{U}} > \tag{7}$$

and, as in the previous case, involves the solution of a linear system of equations. This least squares method, known as "don't care", has the notable property of not suffering from the Gibbs' phenomenon [16]. Its performance on the other hand can be an order of magnitude inferior to the optimum L_∞ filter.

In [21] an L_2 design method was introduced for the linear phase constant weighting function case. This method produces filters that are very close to the L_∞ optimum filters (at worst 3 db inferior) at a computational complexity comparable to the "don't care" method. In the next section we extend this method to the general complex filter case including also general weighting functions. Due to the numerous ways one can take into account the weighting function, it turns out that the proposed extension is not straightforward.

3 Optimization Criterion and Optimum Approximations

To formulate our L_2 approximation problem let us first define the class of frequency responses $d(\omega)$ and weighting functions $w(\omega)$ we are interested in. Let $-\pi = \omega_0 < \omega_1 < \ldots < \omega_{2M-1} = \pi$, be $2M$ distinct points in the interval $[-\pi, \pi]$. Suppose $d(\omega)$ is a complex function defined as

$$d(\omega) = \begin{cases} d_i(\omega) & \omega \in [\omega_{2i}, \omega_{2i+1}], \quad i = 0, \ldots, M-1, \\ g_i(\omega) & \omega \in (\omega_{2i-1}, \omega_{2i}), \quad i = 1, \ldots, M-1, \end{cases} \tag{8}$$

and $w(\omega)$ a real positive function defined as

$$w(\omega)=\begin{cases} w_i(\omega) & \omega\in[\omega_{2i},\omega_{2i+1}], \quad i=0,\ldots,M-1,\\ v_i(\omega) & \omega\in(\omega_{2i-1},\omega_{2i}), \quad i=1,\ldots,M-1,\end{cases} \tag{9}$$

where $d_i(\omega),\ w_i(\omega)$ denote the M parts of the frequency response and weighting function that are known (corresponding either to passbands or stopbands) and $g_i(\omega),\ v_i(\omega)$ the $M-1$ parts corresponding to the transition regions that are unknown. If $\mathcal{U}_i=[\omega_{2i},\omega_{2i+1}],\ i=0,\ldots,M-1$, $\mathcal{T}_i=(\omega_{2i-1},\omega_{2i}),\ i=1,\ldots,M-1$, and $\mathcal{U}=\cup_{i=0}^{M-1}\mathcal{U}_i$ and $\mathcal{T}=\cup_{i=1}^{M-1}\mathcal{T}_i$ then $d(\omega)$ and $w(\omega)$ are known on $\mathcal{U}$ and unknown on $\mathcal{T}$. In other words, region $\mathcal{U}$ is the union of the M closed disjoint intervals $\mathcal{U}_i,\ i=0,\ 1,\ \cdots,\ M-1$ where $d(\omega)$ and $w(\omega)$ are known, while $\mathcal{T}$ is the union of the $M-1$ open disjoint intervals $\mathcal{T}_i,\ i=1,\ 2,\ \cdots,\ M-1$ where $d(\omega)$ and $w(\omega)$ are unknown.

According to our definition of $\mathcal{U}$, the two end intervals $[-\pi,\omega_1]$ and $[\omega_{2(M-1)},\pi]$ correspond to intervals where $d(\omega)$ and $w(\omega)$ are known. Due to the periodicity of the frequency response, we need to assume that *the two end intervals are of the same type*, that is, they are both either passbands or stopbands with the desired response and the weighting function satisfying $d(-\pi)=d(\pi)$ and $w(-\pi)=w(\pi)$. Notice that this constraint can always be satisfied by applying a simple frequency shift in our design problem.

Our goal is to properly exploit the unknown part of the desired response in order to come up with a computationally efficient filter design method that produces filters having performance very close to the L_∞ optimum filters. Since the weighting function is also unknown inside the transition regions, in order to facilitate our design, and simultaneously to ensure the desired monotonicity of the weighting function on each transition interval, we propose to extend $w(\omega)$ on each transition interval $\mathcal{T}_i=(\omega_{2i-1},\omega_{2i})$ by using a simple linear or exponential interpolation scheme of the form $w(\omega)=v_i(\omega)=\alpha_i\omega+\beta_i$ or $w(\omega)=v_i(\omega)=\alpha_i e^{\beta_i\omega}$ respectively. In both cases parameters $\alpha_i,\ \beta_i$ can be uniquely specified by assuring that the resulting $w(\omega)$ is continuous. Specifically, by imposing on each transition interval $\mathcal{T}_i=(\omega_{2i-1},\omega_{2i})$ the necessary continuity constraints, we can easily obtain that the form of $v_i(\omega)$ for linear interpolation is

$$v_i(\omega)=w(\omega_{2i-1})+\frac{w(\omega_{2i})-w(\omega_{2i-1})}{\omega_{2i}-\omega_{2i-1}}(\omega-\omega_{2i-1}),\quad \omega\in\mathcal{T}_i, \tag{10}$$

whereas for the exponential

$$v_i(\omega)=w(\omega_{2i-1})^{\frac{\omega_{2i}-\omega}{\omega_{2i}-\omega_{2i-1}}}\,w(\omega_{2i})^{\frac{\omega-\omega_{2i-1}}{\omega_{2i}-\omega_{2i-1}}},\quad \omega\in\mathcal{T}_i. \tag{11}$$

Both interpolations result in optimum filters that have comparable performance.

For the known parts of the desired response and the weighting function, that is, functions $d_i(\omega),\ w_i(\omega)$ we make the following assumption:

$\mathcal{A}$: The parts $d_i(\omega),\ w_i(\omega)$ of the desired response and the weighting function defined on the closed intervals $\mathcal{U}_i = [\omega_{2i}, \omega_{2i+1}],\ i = 0, \ldots, M-1$, are continuous functions with a piecewise continuous derivative.

In other words *inside* each passband and stopband the desired response and the weighting function are required to be well defined functions up to first order derivatives. This assumption is clearly not very restrictive since most well known filter design problems satisfy it.

3.1 Optimization Criterion

Since the desired response $d(\omega)$ is not defined in the transition regions, by selecting the functions $g_i(\omega)$, we end up with different possibilities for the desired response $d(\omega)$. For each such selection there corresponds an optimum filter defined by (6) that minimizes the weighted mean square error criterion $\|wd - wh\|^2$, where $h(\omega)$ is the filter frequency response defined in (4). It is clear that the filter that minimizes the weighted mean square error will depend on the specific selection of $d(\omega)$, let us therefore denote it as $\mathbf{h}_d$; furthermore the corresponding minimum weighted mean square error will also be a function of $d(\omega)$:

$$\mathcal{E}_0(d) = \|wd - wh_d\|^2, \tag{12}$$

where the optimum filter coefficients $\mathbf{h}_d$ are given by (6).

Using (12), we can now propose a means to optimally define the desired response $d(\omega)$ by further minimizing $\mathcal{E}_0(d)$ with respect to $d(\omega)$, that is,

$$d_o(\omega) = \arg\min_d \mathcal{E}_0(d) = \arg\min_d \|wd - wh_d\|^2. \tag{13}$$

This will also yield the optimum filter coefficients as

$$\mathbf{h}_{d_o} =< w\mathbf{e}, w\mathbf{e} >^{-1}< w\mathbf{e}, wd_o^* > . \tag{14}$$

With the following lemma we have the solution to the optimization problem defined by (13) and (6).

Lemma 1: *The minimization problem defined by (13) and (6) yields as optimum desired response, in the transition regions, $d_o(\omega) = \mathbf{e}^H(\omega)\mathbf{h}_{dc}$ and as optimum filter $\mathbf{h}_o = \mathbf{h}_{dc}$ where $\mathbf{h}_{dc}$ is the optimum don't care filter defined by (7).*

Proof: For any filter $\mathbf{h}$ and any selection $d(\omega)$ we have

$$\|wd - wh\|^2 \geq \|(wd - wh)\mathbb{1}_U\|^2 \geq \|(wd - wh_{dc})\mathbb{1}_U\|^2, \tag{15}$$

where the first inequality comes from the fact that we integrate a nonnegative quantity over a smaller interval and the second from the fact that $\mathbf{h}_{dc}$ is the optimum don't care filter thus minimizing the second quantity. We therefore conclude that the don't care optimum weighted mean square error constitutes a lower bound on the weighted mean square error of any filter for any selection of the desired response. Furthermore this last quantity is independent of the specific selection of $d(\omega)$ since we don't take into consideration the transition regions.

If we now select $d_o(\omega) = \mathbf{e}^H(\omega)\mathbf{h}_{dc}$ in the transition regions, then it is a simple exercise to verify that the corresponding filter from (6) satisfies to $\mathbf{h}_{d_o} = \mathbf{h}_{dc}$ which suggests that the error $d_o(\omega) - h_{d_o}(\omega) = 0$ inside the transition regions therefore attaining the lower bound. ■

The main drawback of the don't care solution is the fact that the resulting optimum desired response is not necessarily continuous, furthermore, don't care optimum filters are known to be 6-10 db inferior to the optimum min-max filters. In order to come up with a computationally efficient filter design method that produces filters with improved characteristics we extend the idea presented in [21] and propose the following optimality criterion

$$\mathcal{E}_k(d) = \|(wd)^{(k)} - (wh_d)^{(k)}\|^2 \tag{16}$$

where superscript "$^{(k)}$" denotes k-th derivative of the corresponding function with respect to the frequency ω. One point that needs to be stressed is the fact that the proposed optimality criterion depends on the integer k we select to use. Notice the special case which arises for $k = 0$. For this case the proposed optimality criterion coincides with the criterion used by the don't care method. However, for $k > 0$ the above mentioned criteria are completely different, and thus from their optimization we are expecting to obtain completely different optimal filters. We are mainly interested in the case $k = 1$, since no significant improvement is observed when employing higher values of k. Our final performance measure is therefore the following:

$$\mathcal{E}_1(d) = \|(wd)' - (wh_d)'\|^2 \tag{17}$$

where $h_d(\omega)$ is the frequency response of the filter $\mathbf{h}_d$ defined in (6). As in the case of $\mathcal{E}_0(\cdot)$, we now propose the following optimization problem

$$d_o(\omega) = \arg\min_d \mathcal{E}_1(d) = \arg\min_d \|(wd)' - (wh_d)'\|^2. \tag{18}$$

The existence of the derivatives in the criterion (17) immediately excludes discontinuous desired responses as having infinite weighted mean squared error. Indeed this is so, because the derivative at a discontinuity is a Dirac function, which is not square-integrable. This suggests that in order to solve the optimization problem (18), we can limit ourselves to desired responses $d(\omega)$ that are (right and left) differentiable yielding a well defined function $\big(w(\omega)d(\omega)\big)'$ (not necessarily continuous). Without loss of generality we can therefore impose the following constraint on $d(\omega)$.

$\mathcal{C}$: The desired response $d(\omega)$ is a continuous function which, at each frequency ω has finite left and right derivatives.

Of course in view of Assumption $\mathcal{A}$ we need to apply Constraint $\mathcal{C}$ only in the transition regions paying special attention to the end points of each such interval. Notice also that there is no problem coming from the weighting function $w(\omega)$ since this function, with the extension proposed in (10) or (11), has well defined right and left derivatives.

Concluding, given that the parts $d_i(\omega)$ of the desired response defined on the intervals $\mathcal{U}_i,\ i = 0,\ 1, \ldots, M-1$ are known and they are satisfying $\mathcal{A}$, we would like to minimize $\mathcal{E}_1(d)$ over all complex functions $g(\omega)$ given by (8) and satisfying constraint $\mathcal{C}$. It is clear that the function $g(\omega)$ needs to be defined only inside the region $\mathcal{T}$ since in the region $\mathcal{U}$ it is already known.

3.2 Optimum Filter and Optimum Desired Response

We will now present a theorem that gives necessary and sufficient conditions for the optimality of the desired response $d_o(\omega)$ and its corresponding filter $\mathbf{h}_{d_o}$.

Theorem 1: *The function $d_o(\omega)$ and its corresponding filter $\mathbf{h}_{d_o}$ defined by (14), solve the optimization problem defined by (18) and (6) if and only if inside each transition region*

$\mathcal{T}_i,\ i = 1, \ldots, M-1$, *the following ordinary differential equation is satisfied:*

$$[w(\omega)(d_o(\omega) - h_{d_o}(\omega))]'' = -w(\omega)\mathbf{e}^H(\omega)\mathbf{p} \tag{19}$$

where

$$\mathbf{p} = < w\mathbf{e}, w\mathbf{e} >^{-1} < (w\mathbf{e})', [w(d_o^* - h_{d_o}^*)]' > . \tag{20}$$

Proof: The proof involves standard variational techniques [10] and is presented in the Appendix. ■

With the help of Theorem 1 we can find a system of linear equations that solves the optimization problem defined by (18). In fact the method we are going to propose provides simultaneously the optimum filter coefficients $\mathbf{h}_{d_o}$ and the optimum desired response $d_o(\omega)$ inside the transition regions of the filter. A point worth mentioning is the fact that when the weighting function $w(\omega)$ is constant then, from (20), the quantity $< (w\mathbf{e})', [w(d_o^* - h_{d_o}^*)]' >$, using integration by parts, is equal to $w < \mathbf{e}'', d_o^* - h_{d_o}^* >= 0$, with the last equality being the result of the orthogonality principle. Therefore for $w(\omega)$ constant, (19) is reduced to the differential equation of Theorem 1 of [21].

Let us now present the unknown variables and the corresponding linear equations needed for the solution of the optimization problem. Notice that we already have introduced two parameter vectors that are inter-related, namely the optimum filter coefficients $\mathbf{h}_{d_o}$ ($2N+1$ unknowns) and the auxiliary vector $\mathbf{p}$ defined in (20) which also contains ($2N+1$) unknowns. If we now integrate twice the differential equation in (19) inside each transition region we obtain

$$w(\omega)d_o(\omega) = w(\omega)\mathbf{e}^H(\omega)\mathbf{h}_{d_o} - \mathbf{f}_i(\omega)^H\mathbf{p} + \mathbf{c}^H(\omega)\mathbf{q}_i,\ \omega \in \mathcal{T}_i,\ i = 1, \ldots, M-1, \tag{21}$$

where $\mathbf{c}(\omega) = [\omega\ 1]^t$; $\mathbf{q}_i$ is a vector containing the two unknown parameters of the solution of the differential equation (19), that is, $\mathbf{q}_i = [q_i^1\ q_i^2]^t$ and finally

$$\mathbf{f}_i(\omega) = \int_{\omega_{2i-1}}^{\omega} \int_{\omega_{2i-1}}^{\tau} w(s)\mathbf{e}(s)ds d\tau. \tag{22}$$

is the double consecutive integration of the vector function $w(\omega)\mathbf{e}(\omega)$ with $\mathbf{e}(\omega)$ defined in (3). The vector function $\mathbf{f}_i(\omega)$ can be easily evaluated when we use either the linear or exponential extension of the weighting function (10) or (11). Notice that with (21) we have introduced M additional parameter vectors $\mathbf{q}_i,\ i = 1, \ldots, M-1$, which corresponds to $2M-2$ additional unknown variables thus raising the total number of unknowns to $4N+2M$. It is clear that we need an equal number of equations in order to produce the solution to the optimization problem.

The necessary equations can be obtained from (14) and (20) using (21) and by imposing continuity on the solution $w(\omega)d_o(\omega)$ of (21) at the two end-points of each transition interval $\mathcal{T}_i = (\omega_{2i-1}, \omega_{2i}),\ i = 1, \ldots, M-1$. Specifically, from (14) and using (21) we obtain a first set of $2N+1$ equations as follows

$$\mathbf{A}\mathbf{h}_{d_o} + \mathbf{B}\mathbf{p} + \sum_{i=1}^{M-1} \mathbf{C}_i \mathbf{q}_i = \mathbf{h}_{\mathcal{U}} \tag{23}$$

where

$$\begin{aligned}
\mathbf{h}_{\mathcal{U}} &= < w\mathbf{e}\mathbb{1}_{\mathcal{U}}, wd^* \mathbb{1}_{\mathcal{U}} > && (24)\\
\mathbf{A} &= < w\mathbf{e}\mathbb{1}_{\mathcal{U}}, w\mathbf{e}\mathbb{1}_{\mathcal{U}} > && (25)\\
\mathbf{B} &= \sum_{i=1}^{M-1} < w\mathbf{e}\mathbb{1}_{T_i}, \mathbf{f}_i \mathbb{1}_{T_i} > && (26)\\
\mathbf{C}_i &= - < w\mathbf{e}\mathbb{1}_{T_i}, \mathbf{c}\mathbb{1}_{T_i} >,\ i = 1, \ldots, M-1. && (27)
\end{aligned}$$

Notice that $\mathbf{h}_{\mathcal{U}}$ is a vector of length $2N+1$, $\mathbf{A}$ and $\mathbf{B}$ are matrices of dimensions $(2N+1) \times (2N+1)$ and $\mathbf{C}_i$ is a matrix of dimensions $(2N+1) \times 2$. Notice also that $\mathbf{h}_{\mathcal{U}}$, $\mathbf{A}$, $\mathbf{B}$ and $\mathbf{C}_i$ are quantities that depend only on known functions integrated over known sets and thus they can be considered given.

Similarly using the definition of $\mathbf{p}$ from (20) and using (21) we obtain $2N+1$ additional equations

$$\mathbf{D}\mathbf{h}_{d_o} + \mathbf{E}\mathbf{p} + \sum_{i=1}^{M-1} \mathbf{F}_i \mathbf{q}_i = \mathbf{p}_{\mathcal{U}} \tag{28}$$

where $\mathbf{p}_{\mathcal{U}}$, is a vector of length $2N+1$, $\mathbf{D}$ and $\mathbf{E}$ are matrices of dimensions $(2N+1) \times (2N+1)$ each and $\mathbf{F}_i$ is a matrix of dimensions $(2N+1) \times 2$ and they are defined as follows

$$\begin{aligned}
\mathbf{p}_{\mathcal{U}} &= < (w\mathbf{e})' \mathbb{1}_{\mathcal{U}}, (wd^*)' \mathbb{1}_{\mathcal{U}} > && (29)\\
\mathbf{D} &= < (w\mathbf{e})' \mathbb{1}_{\mathcal{U}}, (w\mathbf{e})' \mathbb{1}_{\mathcal{U}} > && (30)\\
\mathbf{E} &= < w\mathbf{e}, w\mathbf{e} > + \sum_{i=1}^{M-1} < (w\mathbf{e})' \mathbb{1}_{T_i}, \mathbf{f}_i' \mathbb{1}_{T_i} > && (31)\\
\mathbf{F}_i &= - < (w\mathbf{e})' \mathbb{1}_{T_i}, (\mathbf{c})' \mathbb{1}_{T_i} >,\ i = 1, \ldots, M-1. && (32)
\end{aligned}$$

Notice that all the above defined quantities are depended on known functions integrated over known sets and thus they can be considered given.

Finally, in order to satisfy Constraint $\mathcal{C}$ we need to impose continuity on the solution $w(\omega)d_o(\omega)$ of (21) at the two end points $\omega_{2i-1},\ \omega_{2i}$ of each transition region $\mathcal{T}_i$. Since each end point belongs to the set $\mathcal{U}$ the desired response $d(\omega)$ is known at these frequencies, therefore we obtain the following $2M-2$ equations

$$\mathbf{G}_i\mathbf{h}_{d_o} + \mathbf{H}_i\mathbf{p} + \mathbf{J}_i\mathbf{q}_i = \mathbf{s}_i,\ \ i = 1, \ldots, M-1, \tag{33}$$

where

$$\begin{aligned}
\mathbf{s}_i &= [w(\omega_{2i-1})d(\omega_{2i-1})\ \ w(\omega_{2i})d(\omega_{2i})]^t && (34)\\
\mathbf{G}_i &= [w(\omega_{2i-1})\mathbf{e}(\omega_{2i-1})\ \ w(\omega_{2i})\mathbf{e}(\omega_{2i})]^H && (35)\\
\mathbf{H}_i &= -[\mathbf{f}_i(\omega_{2i-1})\ \ \mathbf{f}_i(\omega_{2i})]^H && (36)\\
\mathbf{J}_i &= [\mathbf{c}(\omega_{2i-1})\ \ \mathbf{c}(\omega_{2i})]^H && (37)
\end{aligned}$$

which constitutes the last set of equation raising the total number to the desired $4N + 2M$. Notice that $\mathbf{s}_i$ is a column vector of size 2, $\mathbf{G}_i$ and $\mathbf{H}_i$ are matrices of dimensions $2 \times (2N + 1)$ each and $\mathbf{J}_i$ is a matrix of dimensions 2×2. All these quantities are depended only on known functions integrated over known sets and thus they can be considered given.

Concluding, in order to solve the minimization problem in (18) we solve the linear system of equations defined by (23), (28) and (33) which yields the optimum filter coefficients $\mathbf{h}_{d_o}$, and the auxiliary quantities $\mathbf{p},\ \mathbf{q}_i,\ i = 1, \ldots, M-1$ simultaneously.

4 Design Examples

Let us now apply our method to two design examples and compare it to other existing filter design techniques. In particular we will apply our method to the design of weighted nearly linear phase lowpass and multiband complex filters; and compare it against the don't care method of [14], [16] and the Complex Remez algorithm of [13] which is included in Matlab as the function *cremez.m*.

A. Design of Nearly Linear Phase Lowpass Complex Filters

Consider the following specifications for the known parts of the desired frequency response

$$d_0(\omega) = 0,\ \omega \in [-1, \ \ -0.18]$$

$$
\begin{aligned}
d_1(\omega) &= e^{-j\frac{4N}{5}\omega}, \ \omega \in [-0.1, \ \ 0.3] \\
d_2(\omega) &= 0, \ \omega \in [0.38, \ \ 1]
\end{aligned}
\tag{38}
$$

Notice that for this case, according to our definition of $d(\omega)$ in (8), $M = 3$ (i.e. the number of disjoint intervals where the desired frequency response of the filter is known) and the $2M = 6$ distinct points $\omega_i, \ i = 0, \ \cdots, \ 5$ in the interval $[-\pi, \ \pi]$ that needed for the definition of regions $\mathcal{U}$ and $\mathcal{T}$ are $-\pi = \omega_0 < \omega_1 = -0.18\pi < \omega_2 = -0.1\pi < \omega_3 = 0.3\pi < \omega_4 = 0.38\pi < \omega_5 = \pi$.

Notice also that the specifications of the ideal response of the filter do not satisfy the hermitian symmetry $d(-\omega) = d^\star(\omega)$ and therefore the resulting filter will have complex coefficients.

Let us also consider that the weighting function equal to 1 and $\sqrt{2}$ in the passband and the stopbands respectively. We extend the weighting function inside the transition regions using the exponential interpolation of (11).

In Table I we present the maximal errors e_p, e_s and in Table II the corresponding group delay error e_τ for the three methods and for different filter lengths. The proposed method performs always better than the don't care method. What is however more interesting is the fact that for filter lengths greater than 101 it also outperfoms the Complex Remez algorithm. It is notable the fact that this performance is obtained with a very low computational cost while Complex Remez, as we said, is computationally demanding and practically useless for lengths exceeding 151. We obtained similar results in all other design examples we considered with different values of the cutoff frequencies as well as with different values of the weighting function.

In Figure 1 we plot the form of the magnitude of the optimum ideal response inside the transition regions, for the case $2N + 1 = 51$. Figure 2 shows the trace of the optimum ideal response. As seen from Figure 2, the circle of radius 1 (half-tone) corresponds to the passband of the ideal response whereas the spirals correspond to the first (dotted) and the second (solid) transition regions respectively. Notice that the proposed method yields a continuous optimum ideal response. In Figures 3 and 4 we present the approximation errors in the magnitude and the group delay for the proposed (solid), the don't care (dashed), and the Complex Remez (half-tone), for the complex filter of length 51.

B. Design of Nearly Linear Phase Multiband Filters

Here we consider the following multiband specifications for the known parts of the desired frequency response

$$
\begin{aligned}
d_0(\omega) &= 0,\ \omega \in\ [-1, \quad -0.7] \\
d_1(\omega) &= \frac{1}{2}e^{-j\frac{4N}{5}\omega}, \omega \in\ [-0.65, \quad -0.4] \\
d_2(\omega) &= 0,\ \omega \in [-0.35, \quad -0.1] \\
d_3(\omega) &= 2e^{-j\frac{4N}{5}\omega}, \omega \in\ [-0.05, \quad 0.3] \\
d_4(\omega) &= e^{-j\frac{4N}{5}\omega}, \omega \in\ [0.35, \quad 0.65] \\
d_5(\omega) &= 0,\ \omega \in [0.7, \quad 1]
\end{aligned} \tag{39}
$$

As in the previous example the resulting filter will have complex coefficients. The known parts of the weighting function are defined as follows

$$
\begin{aligned}
w_0(\omega) &= 10,\ \omega \in\ [-1, \quad -0.7] \\
w_1(\omega) &= 1, \omega \in\ [-0.65, \quad -0.4] \\
w_2(\omega) &= 10,\ \omega \in [-0.35, \quad -0.1] \\
w_3(\omega) &= 1,\ \omega \in\ [-0.05, \quad 0.3] \\
w_4(\omega) &= 5, \omega \in\ [0.35, \quad 0.65] \\
w_5(\omega) &= 10,\ \omega \in [0.7, \quad 1]
\end{aligned} \tag{40}
$$

and as in the previous example, it is extended in the transition regions of the multiband filter using the exponential interpolation of (11).

Table III contains the maximum errors e_p, e_s while Table IV the maximum group delay error e_τ for the three methods. Again for this type of filters our method has better performance than the don't care approach and compares favorably to the Complex Remez but at a considerably lower computational cost. This behaviour is typical for this type of filters.

In Figure 5 we plot the form of the magnitude of the optimum ideal response inside the transition regions, for the case $2N + 1 = 151$. Figure 6 shows the trace of the optimum ideal response. The circles in this figure of radii 0.5, 1 and 2 (half-tone) correspond to the first, second and third passband of the ideal response respectively, whereas the spirals correspond to the five transitions regions existing between the stopbands and passbands of the ideal response, and which are optimally defined by the proposed method. As in the previous example, the proposed method provides again a continuous optimum ideal response $d_o(\omega)$. Finally, in Figures 7 and 8 we present the magnitude (in dB) and the approximation errors in group delay for the proposed (solid), the don't care (dashed), and the Complex Remez (half-tone).

5 Conclusion

We have presented a new L_2 based method for the design of arbitrary FIR digital filters. Minimizing a suitable L_2 measure results in an optimum extension of the ideal response inside the transition regions of the filter. The optimum filter is then obtained as the corresponding minimum weighted mean squared error filter. The complexity of the proposed method is low since it requires the solution of a linear system of equations. In all design examples we carried out, our method always outperformed the don't care approach while, at the same time, it either compared favorably or even outperformed the computationally demanding Complex Remez algorithm. Based on the fact that the performance of the proposed method in designing one dimensional FIR filters is very good, its extension to multidimensional filter design problems is currently being investigated.

Acknowledgment

The author would like to thank the anonymous reviewers for their comments and suggestions which helped to improve the clarity of the presentation.

6 Appendix

Proof of Theorem 1: For the proof of our theorem we are going to use conventional variational techniques. Let therefore $d_o(\omega)$ denote the solution to the minimization problem of (18) with the corresponding filter $\mathbf{h}_{d_o}$ defined by (14) and with frequency response $h_{d_o}(\omega) = \mathbf{e}^H(\omega)\mathbf{h}_{d_o}$. Consider now a perturbation of the optimum response of the form

$$d(\omega) = d_o(\omega) + \epsilon\delta(\omega) \tag{41}$$

where ϵ is a complex parameter. In order for $d(\omega)$ to be in the allowable class of functions we must impose certain constraints on $\delta(\omega)$. First we understand that $\delta(\omega) = 0$ for all frequencies in the passbands and stopbands where the desired response is exactly known. Secondly, because of

Constraint $\mathcal{C}$, the existence of the left and right derivative of $d(\omega)$ ensure that the function $\delta(\omega)$ has a left and right derivative, inside each transition region; furthermore because of continuity of $d(\omega)$ at the end points of each transition interval $\mathcal{T}_i$ we conclude that $\delta(\omega)$ is equal to zero at the end-points of each interval $\mathcal{T}_i$.

Due to the linearity of (6) we conclude that

$$h_d(\omega) = h_{d_o}(\omega) + \epsilon h_\delta(\omega). \tag{42}$$

Substituting this in the definition of our performance measure in (17) yields

$$\mathcal{E}_1(d) = \mathcal{E}_1(d_o) + 2\mathrm{Re}\{\epsilon < [w(d_o - h_{d_o})]', [w(\delta - h_\delta)]' >\} + |\epsilon|^2 \mathcal{E}_1(\delta). \tag{43}$$

Assuming $\delta(\omega)$ fixed and ϵ variable $\mathcal{E}_1(d)$ becomes in fact a function of ϵ of the form $\mathcal{E}_1(\epsilon)$. Since $\mathcal{E}_1(\epsilon)$ exhibits a minimum for $\epsilon = 0$ (remember that $d_o(\omega)$ was assumed to minimize our criterion) we conclude that the partial derivatives

$$\left.\frac{\partial \mathcal{E}_1(\epsilon)}{\partial \epsilon_r}\right|_{\epsilon=0} = \left.\frac{\partial \mathcal{E}_1(\epsilon)}{\partial \epsilon_i}\right|_{\epsilon=0} = 0 \tag{44}$$

where ϵ_r, ϵ_i the real and imaginary parts of ϵ respectively. One can then verify that this leads to

$$< [w(d_o - h_{d_o})]', [w(\delta - h_\delta)]' >= 0. \tag{45}$$

Conversely if $d_o(\omega)$ is such that (45) is true then for every allowable perturbation $\delta(\omega)$ we have

$$\begin{aligned} \mathcal{E}_1(d) &= \mathcal{E}_1(d_o) + 2\mathrm{Re}\{\epsilon < [w(d_o - h_{d_o})]', [w(\delta - h_\delta)]' >\} + |\epsilon|^2 \mathcal{E}_1(\delta) & (46) \\ &= \mathcal{E}_1(d_o) + |\epsilon|^2 \mathcal{E}_1(\delta) \geq \mathcal{E}_1(d_o), & (47) \end{aligned}$$

which suggests that $d_o(\omega)$ is optimum since any candidate $d(\omega)$ can be written as $d(\omega) = d_o(\omega) + \delta(\omega)$ with $\delta(\omega) = d(\omega) - d_o(\omega)$. What is left to show is that (45) is equivalent to the differential equation (19).

We observe that

$$\begin{aligned} &< [w(d_o - h_{d_o})]', [w(\delta - h_\delta)]' > \\ &\quad = \; < [w(d_o - h_{d_o})]', [w\delta]' > - < [w(d_o - h_{d_o})]', [wh_\delta]' > \\ &\quad = \; - < [w(d_o - h_{d_o})]'', w\delta > - < [w(d_o - h_{d_o})]', [w\mathbf{e}^H]'\mathbf{h}_\delta > \end{aligned}$$

$$
\begin{aligned}
&= -<[w(d_o-h_{d_o})]'', w\delta> - <[w(d_o-h_{d_o})]', [w\mathbf{e}^H]'> \mathbf{h}_\delta^* \\
&= -<[w(d_o-h_{d_o})]'', w\delta> - <[w(d_o-h_{d_o})]', [w\mathbf{e}^H]'><w\mathbf{e}^*, w\mathbf{e}^*>^{-1}<w\mathbf{e}^*, w\delta> \\
&= -<[w(d_o-h_{d_o})]''+w\mathbf{p}^t\mathbf{e}^*, w\delta>= -<[w(d_o-h_{d_o})]''+w\mathbf{e}^H\mathbf{p}, w\delta>, \qquad (48)
\end{aligned}
$$

where for the second equality, in the first term, we used integration by parts and in the forth equality we replaced $\mathbf{h}_\delta$ by its equal using (6). Combining (48) with (45) and recalling that the perturbation $\delta(\omega)=0$ for $\omega\in\mathcal{U}$, we conclude that

$$
<[w(d_o-h_{d_o})]''+w\mathbf{e}^H\mathbf{p}, w\delta>=\sum_{i=1}^{M-1}<\left([w(d_o-h_{d_o})]''+w\mathbf{e}^H\mathbf{p}\right)\mathbb{1}_{T_i}, w\delta\mathbb{1}_{T_i}>=0. \qquad (49)
$$

Finally since $\delta(\omega)$ can be arbitrarily selected in each transition region, this leads to

$$
[w(d_o-h_{d_o})]''+w\mathbf{e}^H\mathbf{p}=0,\quad \omega\in\mathcal{T}_i,\ i=1,\ldots,M-1. \qquad (50)
$$

This concludes the proof. ■

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Figure Captions

Figure 1. Optimum ideal magnitude response for the design of a nearly linear phase lowpass complex filter of length 51.

Figure 2. Trace of the optimum ideal response of a nearly linear phase lowpass complex filter of length 51. Passband (half-tone), Transitions regions (solid and dotted).

Figure 3. Magnitude approximation errors for the design of a nearly linear phase lowpass complex filter of length 51. Proposed method (solid), Complex Remez (half-tone), don't care region (dashed).

Figure 4. Group delay approximation errors inside the passband for the design of a nearly linear phase lowpass complex filter of length 51. Proposed method (solid), Complex Remez (half-tone), don't care region (dashed).

Figure 5. Optimum ideal magnitude response for the design of a nearly linear phase multiband complex filter of length 151.

Figure 6. Trace of the optimum ideal response of a nearly linear phase multiband complex filter of length 151. Passbands (half-tone), Transitions regions (solid and dotted).

Figure 7. Magnitude response of a nearly linear phase multiband complex filter of length 151. Proposed method (solid), Complex Remez (half-tone), don't care region (dashed).

Figure 8. Group delay approximation errors inside the passbands for the design of a nearly linear phase multiband complex filter of length 151. Proposed method (solid), Complex Remez (half-tone), don't care region (dashed).

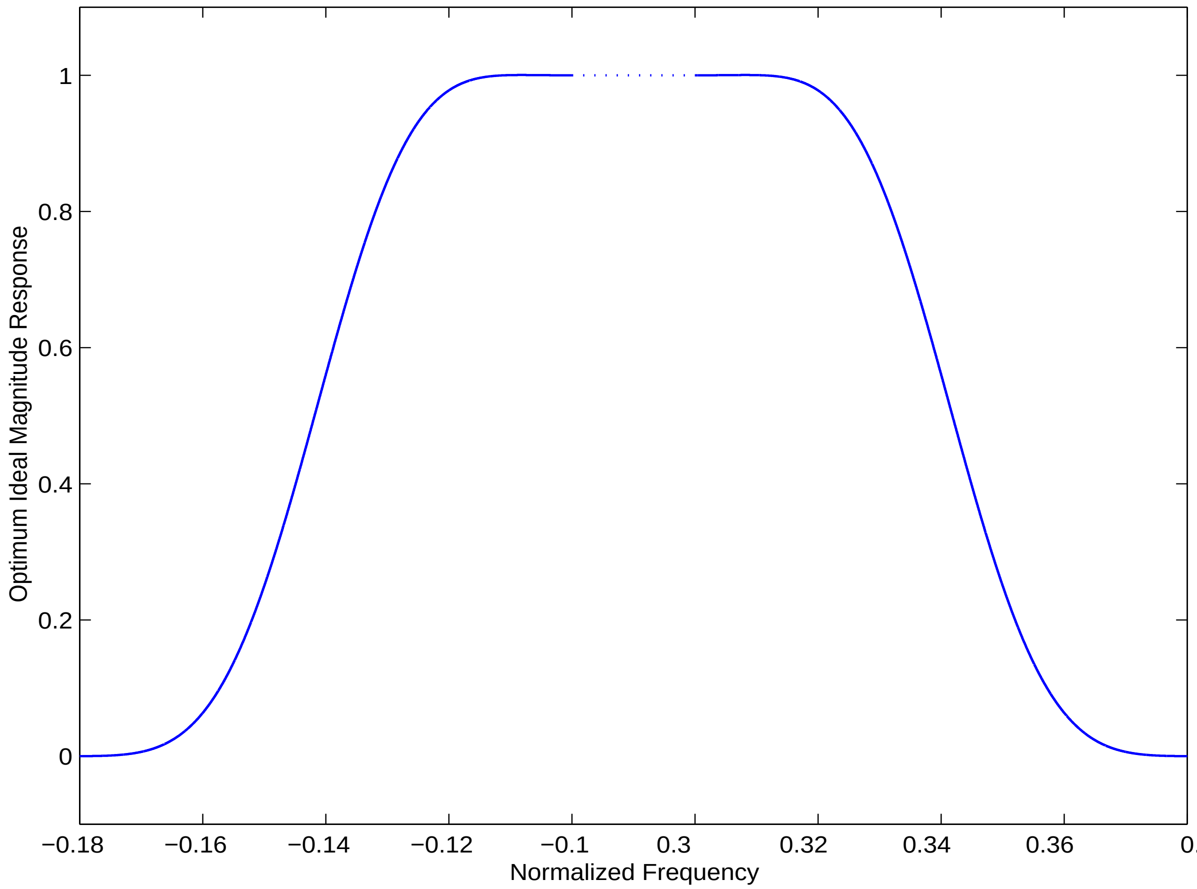

Figure 1. Optimum ideal magnitude response for the design of a nearly linear phase lowpass complex filter of length 51.

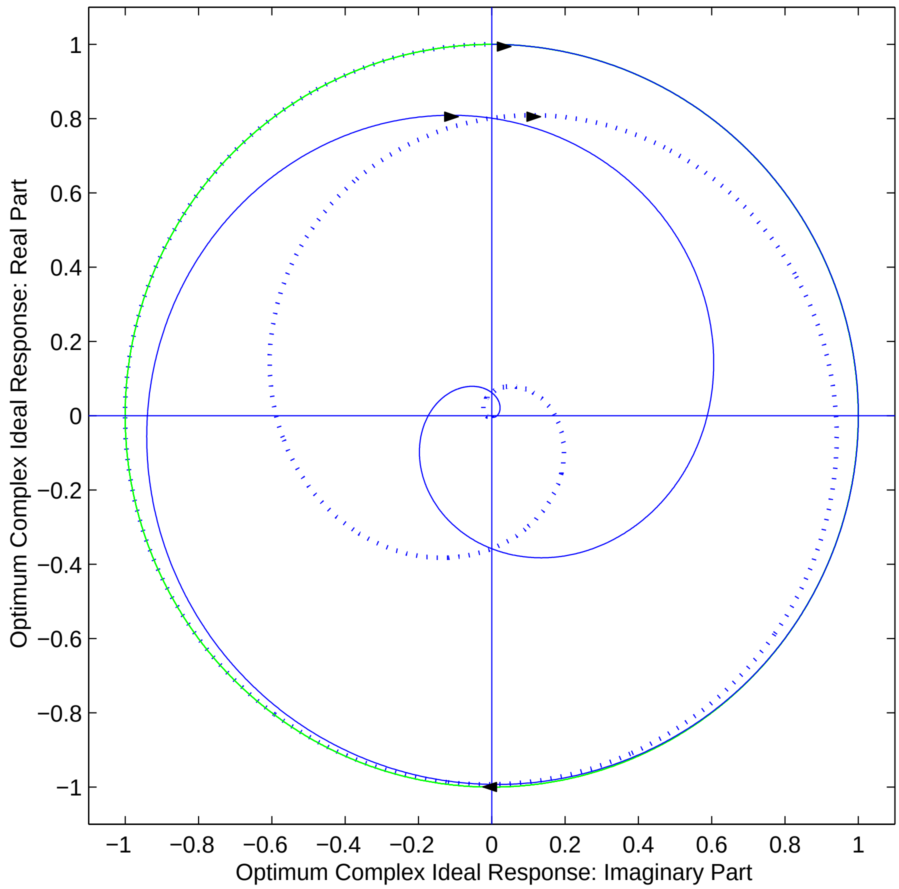

Figure 2. Trace of the optimum ideal response of a nearly linear phase lowpass complex filter of length 51. Passband (half-tone), Transitions regions (solid and dotted).

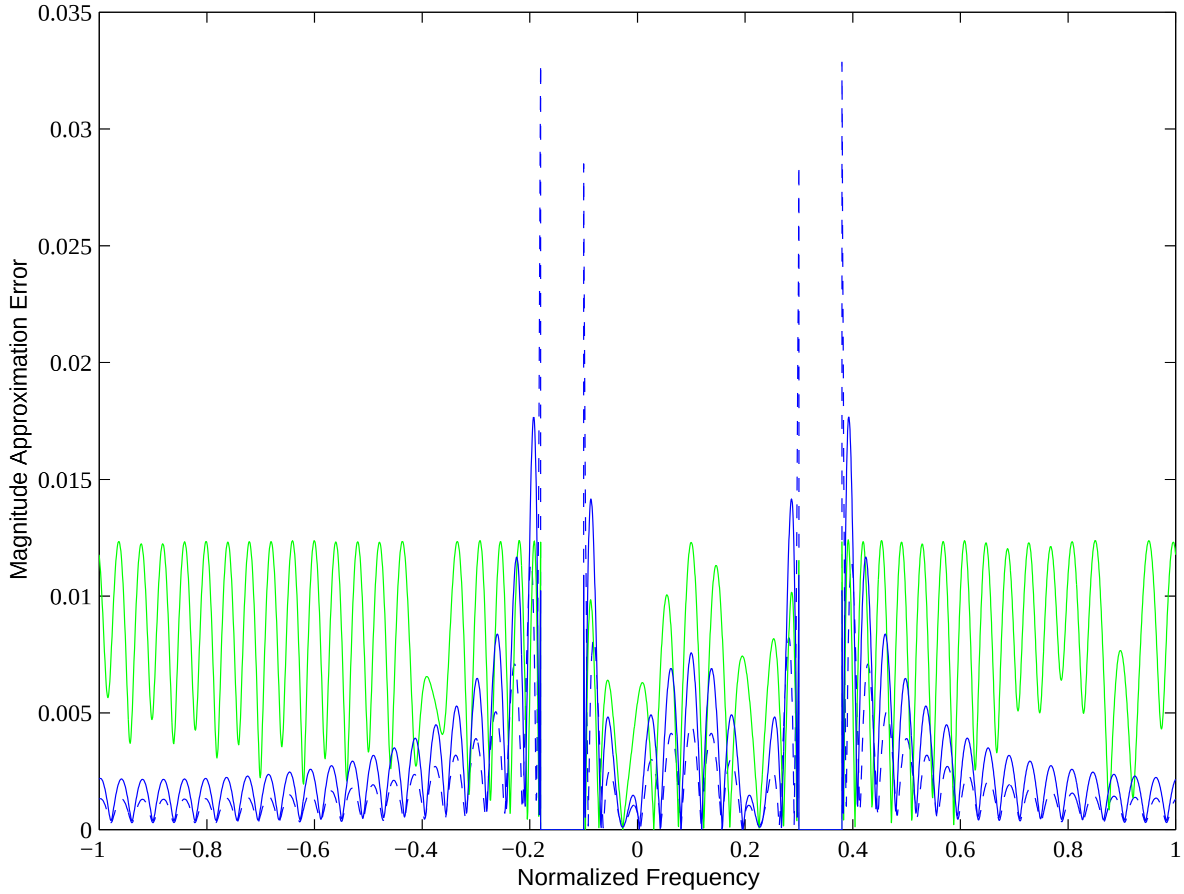

Figure 3. Magnitude approximation errors for the design of a nearly linear phase lowpass complex filter of length 51. Proposed method (solid), Complex Remez (half-tone), don't care region (dashed).

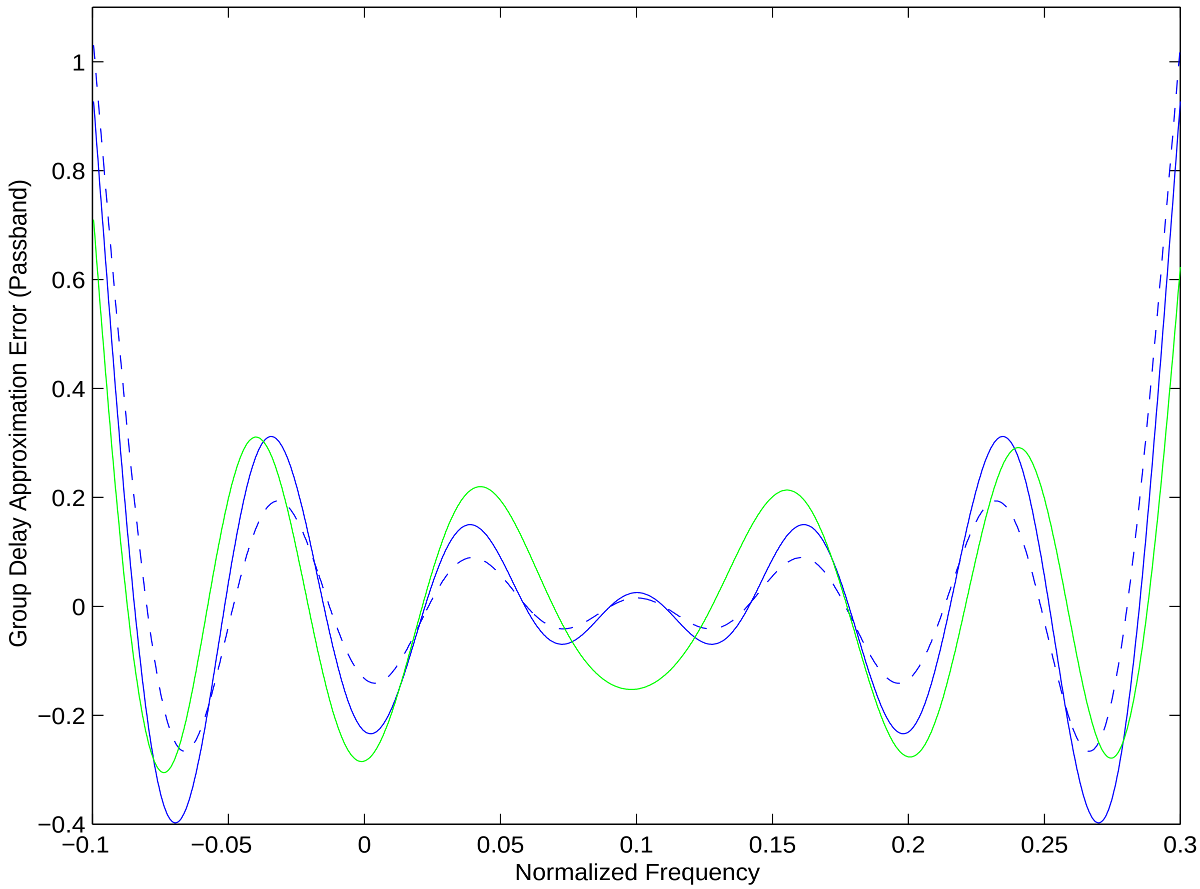

Figure 4. Group delay approximation errors inside the passband for the design of a nearly linear phase lowpass complex filter of length 51. Proposed method (solid), Complex Remez (half-tone), don't care region (dashed).

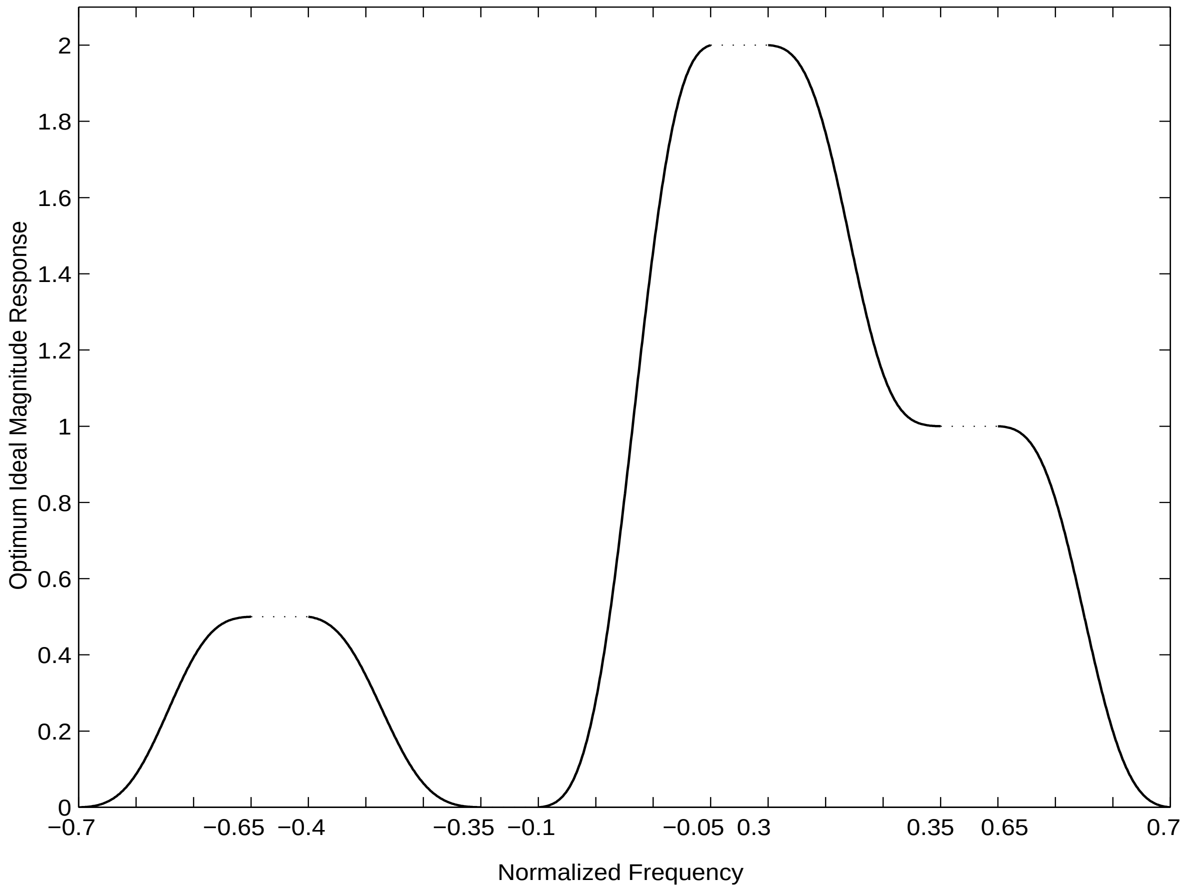

Figure 5. Optimum ideal magnitude response for the design of a nearly linear phase multi-band complex filter of length 151.

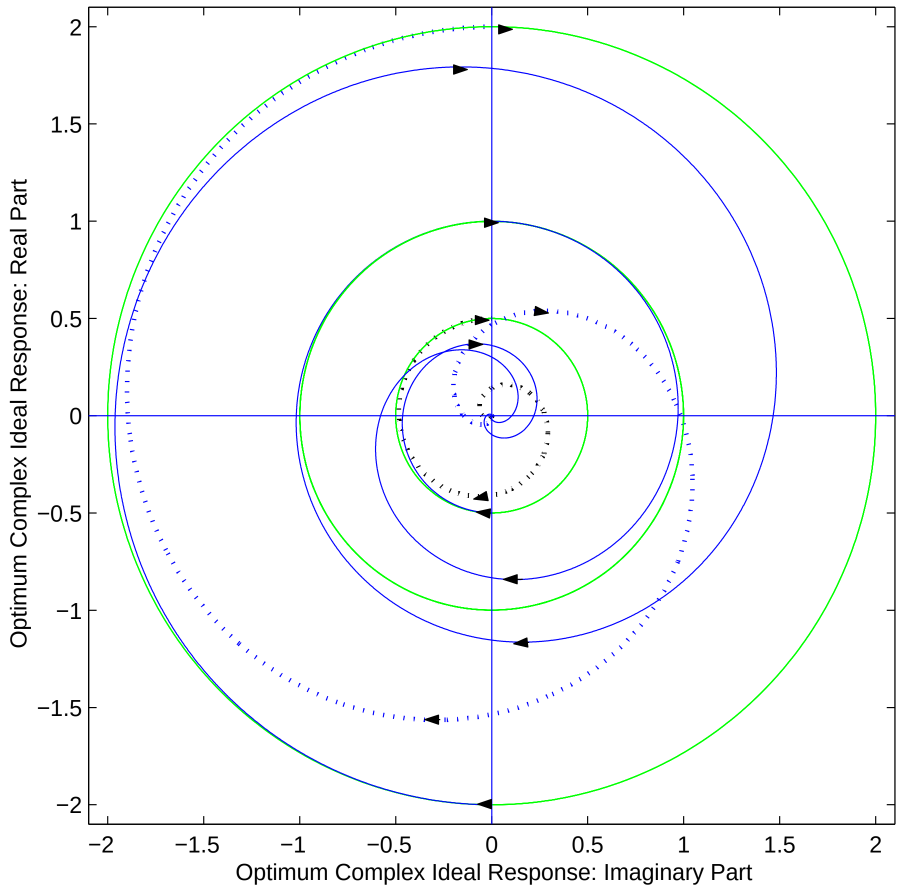

Figure 6. Trace of the optimum ideal response of a nearly linear phase multiband complex filter of length 151. Passbands (half-tone), Transitions regions (solid and dotted).

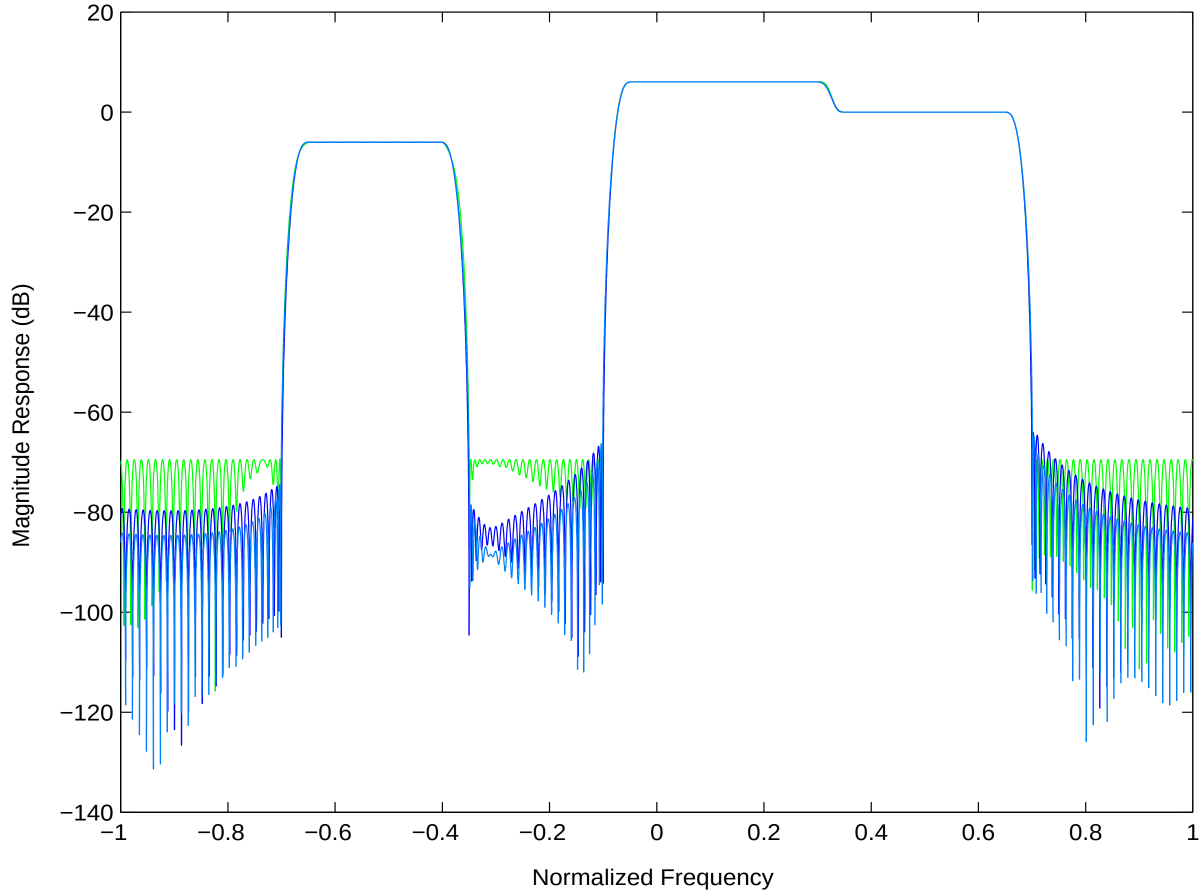

Figure 7. Magnitude response of a nearly linear phase multiband complex filter of length 151.Proposed method (solid), Complex Remez (half-tone), don't care region (dashed).

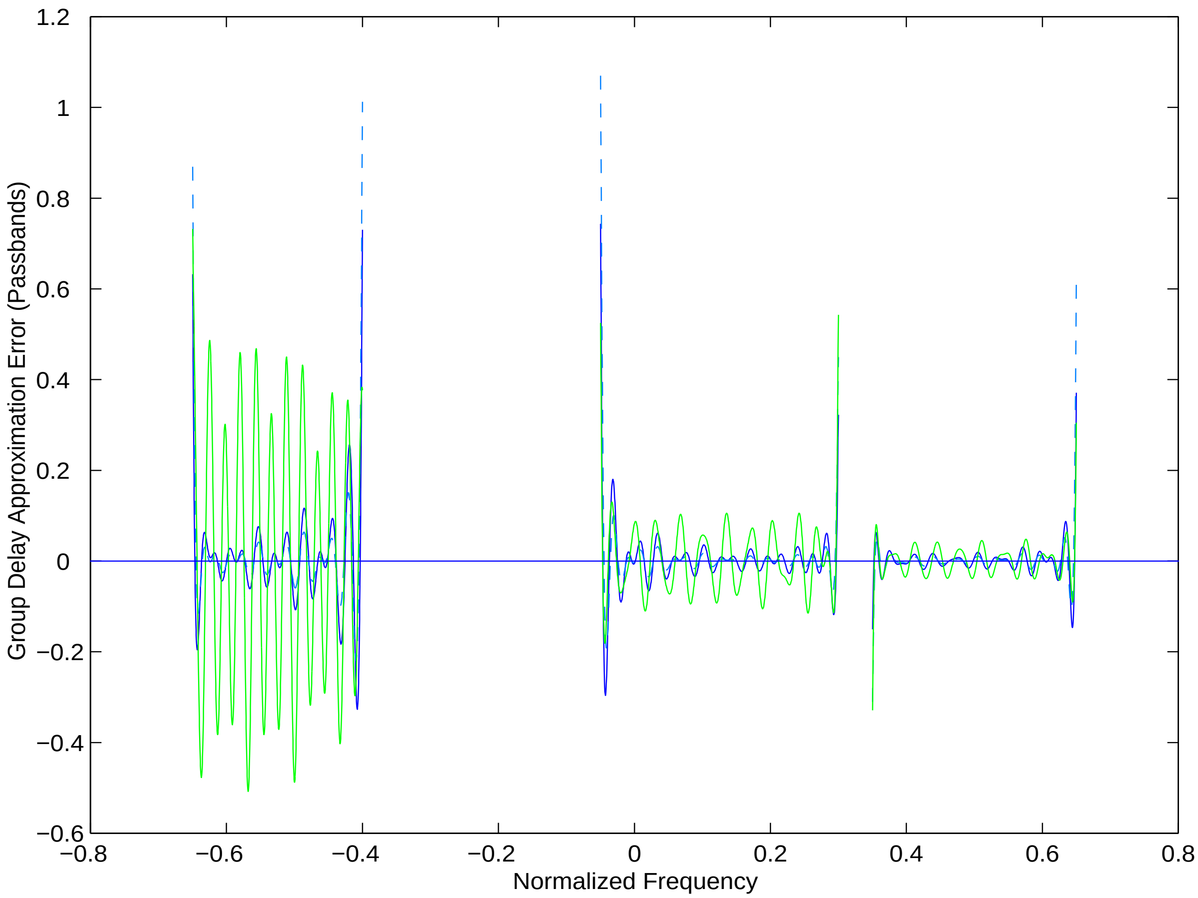

Figure 8. Group delay approximation errors inside the passbands for the design of a nearly linear phase multiband complex filter of length 151. Proposed method (solid), Complex Remez (half-tone), don't care region (dashed).

Table Captions

Table I. Maximum magnitude approximation errors resulted from the weighted design of a nearly linear phase lowpass complex filter by different methods and for different filter lengths.

Table II. Maximum group delay approximation errors resulted from the design of a nearly linear phase lowpass complex filter by different methods and for different filter lengths.

Table III. Maximum magnitude approximation errors resulted from the weighted design of a nearly linear phase multiband complex filter by different methods and for different filter lengths.

Table IV. Maximum group delay approximation errors resulted from the design of a nearly linear phase multiband complex filter by different methods and for different filter lengths.

	Proposed		Don't Care		Complex Remez	
$2N+1$	e_p	e_s	e_p	e_s	e_p	e_s
51	1.42×10^{-2}	1.77×10^{-2}	2.85×10^{-2}	3.29×10^{-2}	1.23×10^{-2}	1.24×10^{-2}
61	5.57×10^{-3}	9.60×10^{-3}	1.18×10^{-2}	1.83×10^{-2}	5.67×10^{-3}	5.75×10^{-3}
71	2.59×10^{-3}	4.87×10^{-3}	5.74×10^{-3}	9.62×10^{-3}	3.69×10^{-3}	3.73×10^{-3}
81	1.06×10^{-3}	2.70×10^{-3}	2.51×10^{-3}	5.75×10^{-3}	1.60×10^{-3}	1.63×10^{-3}
91	6.90×10^{-4}	1.26×10^{-3}	7.76×10^{-4}	2.86×10^{-3}	1.97×10^{-3}	1.98×10^{-3}
101	3.27×10^{-4}	7.16×10^{-4}	3.85×10^{-4}	1.76×10^{-3}	5.83×10^{-4}	6.29×10^{-4}
111	2.12×10^{-4}	3.35×10^{-4}	1.32×10^{-4}	8.75×10^{-4}	2.27×10^{-4}	4.43×10^{-4}
121	1.26×10^{-4}	1.93×10^{-4}	7.83×10^{-5}	5.13×10^{-4}	4.88×10^{-4}	1.53×10^{-4}
131	6.55×10^{-5}	9.75×10^{-5}	5.49×10^{-5}	2.71×10^{-4}	4.03×10^{-4}	7.04×10^{-5}
141	4.35×10^{-5}	5.01×10^{-5}	5.07×10^{-5}	1.43×10^{-4}	8.33×10^{-5}	6.01×10^{-5}
151	2.07×10^{-5}	2.77×10^{-5}	2.96×10^{-5}	8.25×10^{-5}	7.96×10^{-5}	1.52×10^{-5}

Table I. Maximum magnitude approximation errors resulted from the weighted design of a nearly linear phase lowpass complex filter by different methods and for different filter lengths.

	Proposed	Don't Care	Complex Remez
$2N+1$	e_τ	e_τ	e_τ
51	9.27×10^{-1}	1.03×10^{0}	7.10×10^{-1}
61	6.84×10^{-1}	8.54×10^{-1}	6.99×10^{-1}
71	5.42×10^{-1}	7.39×10^{-1}	5.13×10^{-1}
81	3.23×10^{-1}	4.87×10^{-1}	3.43×10^{-1}
91	2.31×10^{-1}	3.91×10^{-1}	2.66×10^{-1}
101	1.35×10^{-1}	2.48×10^{-1}	1.46×10^{-1}
111	8.13×10^{-2}	1.66×10^{-1}	8.06×10^{-2}
121	5.04×10^{-2}	1.12×10^{-1}	5.00×10^{-2}
131	2.59×10^{-2}	6.27×10^{-2}	2.93×10^{-2}
141	1.62×10^{-2}	4.28×10^{-2}	1.61×10^{-2}
151	8.00×10^{-3}	2.27×10^{-2}	1.04×10^{-2}

Table II. Maximum group delay approximation errors resulted from the design of a nearly linear phase lowpass complex filter by different methods and for different filter lengths.

	Proposed		Don't Care		Complex Remez	
$2N+1$	e_p	e_s	e_p	e_s	e_p	e_s
51	4.87×10^{-1}	3.55×10^{-1}	5.67×10^{-1}	6.15×10^{-1}	2.50×10^{-1}	2.51×10^{-1}
61	3.04×10^{-1}	2.04×10^{-1}	3.78×10^{-1}	4.01×10^{-1}	1.53×10^{-1}	1.56×10^{-1}
71	1.90×10^{-1}	1.58×10^{-1}	2.38×10^{-1}	3.06×10^{-1}	1.00×10^{-1}	1.01×10^{-1}
81	9.43×10^{-2}	1.08×10^{-1}	1.51×10^{-1}	2.02×10^{-1}	6.31×10^{-2}	6.37×10^{-2}
91	5.21×10^{-2}	6.91×10^{-2}	1.05×10^{-1}	1.33×10^{-1}	4.44×10^{-2}	4.49×10^{-2}
101	3.10×10^{-2}	4.67×10^{-2}	6.51×10^{-2}	9.13×10^{-2}	2.70×10^{-2}	2.73×10^{-2}
111	1.84×10^{-2}	3.43×10^{-2}	3.99×10^{-2}	6.97×10^{-2}	1.80×10^{-2}	1.80×10^{-2}
121	1.08×10^{-2}	2.09×10^{-2}	2.46×10^{-2}	4.35×10^{-2}	1.13×10^{-2}	1.14×10^{-2}
131	7.23×10^{-3}	1.31×10^{-2}	1.47×10^{-2}	2.88×10^{-2}	7.84×10^{-3}	7.88×10^{-3}
141	4.33×10^{-3}	8.99×10^{-3}	9.04×10^{-3}	2.06×10^{-2}	5.03×10^{-3}	5.12×10^{-3}
151	3.51×10^{-3}	6.25×10^{-3}	5.58×10^{-3}	1.50×10^{-2}	3.35×10^{-3}	3.39×10^{-3}

Table III. Maximum magnitude approximation errors resulted from the weighted design of a nearly linear phase multiband complex filter by different methods and for different filter lengths.

	Proposed	Don't Care	Complex Remez
$2N+1$	e_τ	e_τ	e_τ
51	3.76×10^{0}	3.34×10^{0}	3.27×10^{0}
61	3.60×10^{0}	4.23×10^{0}	8.05×10^{0}
71	3.01×10^{0}	3.05×10^{0}	7.72×10^{0}
81	4.37×10^{0}	4.04×10^{0}	7.27×10^{0}
91	3.02×10^{0}	3.20×10^{0}	6.53×10^{0}
101	2.23×10^{0}	2.34×10^{0}	2.10×10^{0}
111	3.05×10^{0}	3.53×10^{0}	5.65×10^{0}
121	1.76×10^{0}	1.96×10^{0}	2.14×10^{0}
131	1.78×10^{0}	2.36×10^{0}	3.11×10^{0}
141	1.23×10^{0}	1.41×10^{0}	2.42×10^{0}
151	7.43×10^{-1}	1.07×10^{0}	7.32×10^{-1}

Table IV. Maximum group delay approximation errors resulted from the design of a nearly linear phase multiband complex filter by different methods and for different filter lengths.