STEADY STATE KALMAN FILTER: A NEW APPROACH

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ABSTRACT

In this paper a new approach for the steady state Kalman Filter implementation is proposed. The method is faster than the classical one; this is very important due to the fact that, in most real-time applications, it is essential to obtain the estimate in the shortest possible time.

1. INTRODUCTION

Estimation plays an important role in many fields of science. The discrete time Kalman Filter [1] is the most well known algorithm that solves the estimation/filtering problem. Many real world problems have been successfully solved using the Kalman Filter ideas; filter applications to aerospace industry, chemical process, communication systems design, control, civil engineering, filtering noise from 2-dimensional images, pollution prediction, power systems are mentioned in [2]. In this paper we propose a method to implement the steady state Kalman Filter. It is established that the method is faster than the classical one.

2. STEADY STATE KALMAN FILTER

The steady state estimation problem arises in linear estimation and is associated with time invariant systems described by the following state space equations:

$$x(k+1) = Fx(k) + w(k) \quad (1)$$

$$z(k+1) = Hx(k+1) + v(k+1) \quad (2)$$

where x(k) is the n-dimensional state vector at time k, z(k) is the m-dimensional measurement vector, F is the nxn system transition matrix, H is the mxn output matrix, {w(k)} and {v(k)} are Gaussian zero-mean white and uncorrelated random processes, Q and R are the plant and measurement noise covariance matrices respectively, x(0) is a Gaussian random process with mean x_0 and covariance P_0 and x(0), {w(k)} and {v(k)} are independent.

The filtering/estimation problem is to produce an estimate at time L of the state vector using measurements till time L, i.e. the aim is to use the measurements set {z(1),…,z(L)} in order to calculate an estimate value x(L/L) of the state vector x(L). The discrete time Kalman Filter [1] is summarized in the following:

Kalman Filter (KF)

$$x(k+1/k) = Fx(k/k) \quad (3)$$

$$P(k+1/k) = FP(k/k)F^T + Q \quad (4)$$

$$K(k+1) = P(k+1/k)H^T[HP(k+1/k)H^T + R]^{-1} \quad (5)$$

$$x(k+1/k+1) = x(k+1/k) + K(k+1)[z(k+1) - Hx(k+1/k)] \quad (6)$$

$$P(k+1/k+1) = P(k+1/k) - K(k+1)HP(k+1/k) \quad (7)$$

for k=0,1,… with initial conditions $x(0/0)=x_0$ and $P(0/0)=P_0$

It is well known [2] for time invariant systems that if the signal process model is asymptotically stable (i.e. all eigenvalues of F lie inside the unit circle: $|\lambda_i(F)| < 1$), then there exists a steady state value $\overline{P_e}$ of the estimation error variance P(k/k), which is reached at time k=T where ||P(T/T)-P(T-1/T-1)||< ε and ε is a small positive real number.
The corresponding discrete time Riccati equation [2], [3] results from equations (4), (5) and (7) has as follows:

$$P(k+1/k) = FP(k/k-1)F^T + Q - FP(k/k-1)H^T[HP(k/k-1)H^T + R]^{-1}HP(k/k-1)F^T \quad (8)$$

We are able to calculate the steady state prediction error variance $\overline{P_p}$ by off-line solving the Riccati equation (8). From equation (5) we derive the following relation between the steady state prediction error variance $\overline{P_p}$ and the steady state gain $\overline{K}$:

$$\overline{K} = \overline{P_p}H^T[H\overline{P_p}H^T + R]^{-1} \quad (9)$$

In the steady state case, the resulting discrete time Steady State Kalman Filter has as follows:

Steady State Kalman Filter (SSKF)

$$x(T+k/T+k) = Ax(T+k-1/T+k-1) + \overline{K}z(T+k) \text{ for k=1,2,...} \quad (10)$$

where

$$A = F - \overline{K}HF \quad (11)$$

3. PROPOSED STEADY STATE KALMAN FILTER

In this section, a new approach for the steady state Kalman Filter implementation is presented. The method is based on implementing the steady state Kalman Filter calculations in a different order than the classical method does.
Using equation (10), we have:

$$x(T+k/T+k) = A^k x(T/T) + \sum_{j=1}^{k} c(j)z(T+j) \text{ for some k: k} \in \{1,2,\dots\} \quad (12)$$

where

$$c(j) = A^{k-j}\overline{K}\text{, j=1,…,k} \quad (13)$$

Remarks.

1. This Steady State Kalman Filter implementation requires the Kalman Filter implementation for k=0,…,T-1 in order to calculate the estimate x(T/T).
2. The steady state prediction error variance is calculated by off-line solving the corresponding discrete time Riccati equation [2]-[6]. Then, the steady state gain and the matrix A are calculated off-line using (9) and (11). Finally, the coefficients c(j) in (13) as well as the factor A^k in (12) are calculated off-line.

At this point we observe that matrix A has the following important property:

$$A^k \to 0 \text{ as } k \to \infty \quad (14)$$

In fact, it is well known [2] (see pp.77) that:

$$\text{if } |\lambda_i(F)| < 1\text{, then } |\lambda_i(F - F\overline{K}H)| < 1 \quad (15)$$

From matrix theory, it is given that:

$$\lambda_i(X \cdot Y) = \lambda_i(Y \cdot X)\text{, for X and Y nxn matrices} \quad (16)$$

Then, we have:
$$\lambda_i(A) = \lambda_i(F - \overline{K}HF) = \lambda_i((I - \overline{K}H)F) = \lambda_i(F(I - \overline{K}H)) = \lambda_i(F - F\overline{K}H) \quad (17)$$
Thus, we have:
$$\text{if } |\lambda_i(F)| < 1\text{, then } |\lambda_i(A)| < 1 \quad (18)$$

Due to the computer accuracy, the property of matrix A in (14) leads to the following conclusion: "if the spectral radius of A is less than 1, then the computed powers of A can be expected to converge to zero" [7] (see rule of thumb pp. 356). Thus, there exists some l, such that:
$$A^l \neq 0 \text{ and } A^{l+i} = 0\text{, i=1,2,…} \quad (19)$$

Furthermore, after some algebra, we have:
$$x(T+(\nu+1)l/T+(\nu+1)l) = A^l x(T+\nu l/T+\nu l) + \sum_{j=1}^{l} c(j)z(T+\nu l+j)\text{, for } \nu\text{=0,1,…} \quad (20)$$
where there exists some l as in (19) and
$$c(j) = A^{l-j}\overline{K}\text{, j=1,…,l} \quad (21)$$
Remarks.
1. This Steady State Kalman Filter implementation requires the Kalman Filter implementation for k=0,…,T-1 in order to calculate the estimate x(T/T).
2. The steady state prediction error variance is calculated by off-line solving the corresponding discrete time Riccati equation [2]-[6]. Then, the steady state gain and the matrix A are calculated off-line using (9) and (11). Finally, l in (19) and the coefficients c(j) in (21) are calculated off-line. Note that this implementation can be used to obtain estimates per l lags.

Finally, from equations (12) and (13) we have:
$$x(T+l+k/T+l+k) = A^{l+1}x(T+k-1/T+k-1) + \sum_{j=k}^{l+k} A^{l+k-j}\overline{K}z(T+j) \quad (22)$$
Assuming from (19) that $A^{l+1} = 0$, the following Proposed Steady State Kalman Filter implementation is derived:

Proposed Steady State Kalman Filter (PSSKF)
$$x(T+l+k/T+l+k) = \sum_{j=k}^{l+k} c(j)z(T+j) \text{ for some k: k} \in \{1,2,\ldots\} \quad (23)$$
where there exists some l as in (19) and
$$c(j) = A^{l+k-j}\overline{K}\text{, j=k,…,l+k} \quad (24)$$
Remarks.
1. There is no need for any previous estimates calculation. The calculation of the steady state estimate x(L) at some time L requires the use of the subset of l+1 previous time measurements {z(L-l),…,z(L)}.
2. The steady state prediction error variance as well as the time k=T where the steady state solution is reached, are calculated by off-line solving the corresponding discrete time Riccati equation [2]-[6]. Then, the steady state gain and the matrix A are calculated off-

line using (9) and (11). Finally, l in (19) and the coefficients c(j) in (24) are calculated off-line.

4. COMPUTATIONAL REQUIREMENTS

The computational analysis is based on the analysis in [6]. The following result was used: scalar addition and multiplication operations are involved in matrix manipulation operations, which are needed for the implementation of SSKF and PSSKF (equations (10) and (23)). We define t_{op} as the time needed to calculate the sum of two scalar operands. Table 1 summarizes the calculation burden of matrix operations needed for the implementation of the algorithms.

Table 1. Calculation burden of matrix operations

matrix operation	multiplications	additions	calculation burden
$(nxm)*(mxk)$	nmk	$n(m-1)k$	$2nmk-nk$
$(nx1)+(nx1)$	-	n	n

In order to compare the algorithms we assume that we compute the estimate value x(L/L) of the state vector x(L) at some time L=T+l+s, where T is the time the steady state solution is reached, l as in (19) and s≥1. We define the computation time of each algorithm (SSKF and PSSKF) as the calculation burden required for the on-line calculations of each algorithm multiplied by t_{op} (the calculation burden of all off-line calculations is not taken into account):

$$t(SSKF) = cb(SSKF)t_{op} = \{cb(KF)T + (2n^2 + 2nm - n)(l+s)\}t_{op} \quad (25)$$

$$t(PSSKF) = cb(PSSKF)t_{op} = \{2nm - n + 2nml\}t_{op} \quad (26)$$

where cb(KF) is the per recursion calculation burden needed for the implementation of the Kalman Filter (equations (3)-(7)). Note that cb(PSSKFM) is constant (it depends on the state and measurement vector dimensions, as well as on the off-line calculated l; it does not depend on s).

We define the time improvement from SSKF to PSSKF as:

$$ti(SSKF/PSSKF) = \frac{t(SSKF)}{t(PSSKF)} = \frac{cb(SSKF)}{cb(PSSKF)} = \frac{cb(KF)T + (2n^2 + 2nm - n)(l+s)}{2nm - n + 2nml} \quad (27)$$

From (27) we conlude that PSSKF is faster than SSKF:

$$ti(SSKF/PSSKF) > 1 \quad (28)$$

and

$$ti(SSKF/PSSKF) \to \infty \text{ as } s \to \infty \quad (29)$$

5. SIMULATION RESULTS

A simple scalar case (n=1 and m=1) is assumed in the following example, where F=0.8, H=1, Q=10, R=100, P(0/0)=1 and x(0/0)=0. The steady state gain is $\overline{K} = 0.174854$, the steady state solution is reached at time T=21, $A = F - \overline{K}HF = 0.660117$ and l=251.

The calculated estimates using the proposed algorithm PSSKF are equivalent to the calculated estimates using the classical algorithm SSKF. The proposed algorithm PSSKF is faster than the classical algorithm SSKF:

$$ti(SSKF/PSSKF) > 2 \quad (30)$$

as shown in Figure 1, where the time improvement from SSKF to PSSKF is plotted.

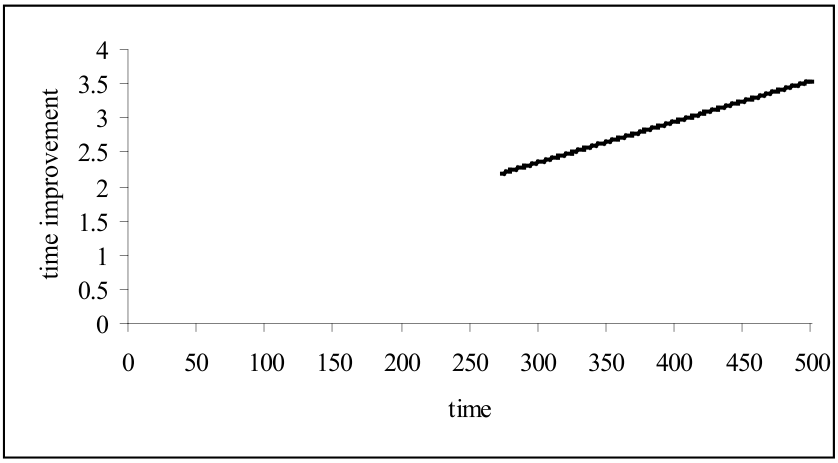

Figure 1. Time improvement from SSKF to PSSKF

6. CONCLUSIONS

A new approach for the steady state Kalman Filter is presented in this paper. The method is based on implementing the Steady State Kalman Filter equations in a different way than the classical algorithm does and taking advantage of the finite computer precision. The classical and the proposed filters are equivalent with respect to their behavior. It was pointed out that the proposed algorithm is faster than the classical one.

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