Design of Two-Dimensional Zero Phase FIR Fan Filters Via the McClellan Transform

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Abstract —**In this paper we present a design method of 2-D zero phase finite impulse response (FIR) fan filters with quadrantal symmetry, via the McClellan transform. We give conditions that the coefficients of the McClellan transform must satisfy, in order to avoid the scaling of the transform. The proposed design method satisfies these conditions. Finally we extend our design method to the design of general shape 2-D zero phase FIR fan filters.**

I. Introduction

THE design of two-dimensional (2-D) digital filters has been of growing interest over the last years. This is due to the variety of applications in fields such as image processing, medical diagnosis, planetary physics, industrial inspection, radar, sonar, seismic and geophysical data processing, pattern recognition, robot vision and weather prediction [1]–[6].

A very important class of 2-D filters is the class of "fan filters". The fan filter is a 2-D filter which has long been used to process geoseismic data. It has the ability to pass seismic events whose apparent velocities on the earth's surface fall within a wedge shaped region in the frequency wavenumber plane. Design approaches for 2-D finite impulse response (FIR) and 2-D infinite impulse response (IIR) fan filters can be broadly classified into two categories:

i) based on transformations of one-dimensional (1-D) filters [8]–[11], [21]–[24];
ii) based on direct min–max and L_p optimization techniques [16]–[20], [25].

The design approaches of the IIR filters [1]–[6], [16]–[24] are generally more complicated than the corresponding approaches for FIR filters, since stability constraints must also be considered.

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In this paper we will use a design method which falls into category (i). Category (i) has the property to divide the design problem into two decoupled design subproblems. Namely, the selection of a high order 1-D filter and the selection of a low order 2-D transform, thereby allowing high order 2-D designs to be obtained with small computational effort. Furthermore, these filters can often be designed to be optimal in the Chebychev sense.

Since the two subproblems are decoupled and since there exist very powerful methods for designing 1-D filters; in this paper we will be concentrated only on the design of the 2-D to 1-D transform. Specifically for the 2-D to 1-D transform we will be using the McClellan transform. The McClellan transform has been proved to be a very useful tool for the design of 2-D digital filters [8], [9], [14], [15]. In general the coefficients of the McClellan transform are computed using optimization techniques [9], [14], [15]. These techniques require a large computational effort. Thus the need for approximate solutions is necessary. In [10] such an approximation technique is presented which results in simple formulas for fast calculation of the McClellan transform coefficients.

The goal of this paper is to present the design of 2-D zero phase FIR fan filters with arbitrary inclination θ by using the McClellan transform. A fast approximate method is proposed. This method derives filters that are very close to the ideal specifications.

This paper is organized as follows. Section II contains a brief presentation of the McClellan transform and how it can be applied to the design of 2-D filters. In Section III we give necessary and sufficient conditions that the McClellan transform must satisfy in order to be applicable to the design of fan filters. In Section IV we present our design method. In Section V we apply our method to the design of 2-D zero phase FIR fan filters with quadrantal symmetry, and we extend it to general shape fan filters without quadrantal symmetry. Finally, in Section VI we present the conclusions.

II. The McClellan Transform

In many signal processing applications we are interested in zero phase filters. Thus consider a 1-D filter with frequency response $\hat{G}(e^{j\omega})$ that has the property $\hat{G}(e^{j\omega}) =$

$G(\cos(\omega))$, i.e., the impulse response is symmetric. The McClellan transform method uses the relation

$$\cos(\omega) = f(\omega_1, \omega_2) \tag{2.1}$$

where ω is the 1-D frequency and (ω_1, ω_2) is the 2-D frequency pair. Specifically the McClellan transform defines $f(\omega_1, \omega_2)$ as follows:

$$\cos(\omega) = f_{\mathrm{GM}}(\omega_1, \omega_2) = \sum_{i=0} \sum_{j=0} t_{ij} \cos(i\omega_1) \cos(j\omega_2) + \sum_{k=1} \sum_{l=1} s_{kl} \sin(k\omega_1) \sin(l\omega_2) \tag{2.2}$$

(GM for Generalized McClellan transform). Notice from (2.1) that a necessary condition $f_{\mathrm{GM}}(\omega_1, \omega_2)$ must satisfy is

$$-1 \leqslant f_{\mathrm{GM}}(\omega_1, \omega_2) \leqslant 1, \qquad \text{for } \omega_1, \omega_2 \in [-\pi, \pi]. \tag{2.3}$$

Usually we like to have points (ω_1, ω_2) that satisfy the equalities in (2.3), in order to cover the whole 1-D band.

In this paper we will use the original McClellan transform which has quadrantal symmetry and is given by

$$\begin{aligned} \cos(\omega) &= f_{\mathrm{OM}}(\omega_1, \omega_2) \\ &= t_{00} + t_{10} \cos(\omega_1) + t_{01} \cos(\omega_2) \\ &\quad + t_{11} \cos(\omega_1) \cos(\omega_2) \end{aligned} \tag{2.4}$$

(OM for Original McClellan transform). Substituting in $G(\cos(\omega))$ the $\cos(\omega)$ from (2.4), the 1-D frequency response $\hat{G}(e^{j\omega})$ is transformed into a 2-D frequency response $\hat{H}(e^{j\omega_1}, e^{j\omega_2})$, which has quadrantal symmetry and thus satisfies $\hat{H}(e^{j\omega_1}, e^{j\omega_2}) = H(\cos(\omega_1), \cos(\omega_2))$. Notice that if the 1-D zero phase filter is FIR with impulse response of extend $(2M+1)$, then the impulse response $h(n,m)$, (that corresponds to $\hat{H}(e^{j\omega_1}, e^{j\omega_2})$), is of extend $(2M+1) \times (2M+1)$ and satisfies the following symmetry conditions:

$$h(2M-k, m) = h(k, m), \qquad k = 0, 1, \cdots, M-1 \tag{2.5a}$$

$$h(n, 2M-l) = h(n, l), \qquad l = 0, 1, \cdots, M-1. \tag{2.5b}$$

To a given $\omega \in [0, \pi]$ corresponds a curve in the (ω_1, ω_2) plane. Along this curve the transformed frequency response $\hat{H}(e^{j\omega_1}, e^{j\omega_2})$ is constant equal to the value of the 1-D frequency response $\hat{G}(e^{j\omega})$ at the point ω. As ω varies, a family of contours is generated which completely describes the transformed frequency response. The contours, or iso-potentials, in the (ω_1, ω_2) plane corresponding to the equation

$$\hat{H}(e^{j\omega_1}, e^{j\omega_2}) = c_1 \tag{2.6}$$

are the same with the contours corresponding to the equation

$$f_{\mathrm{OM}}(\omega_1, \omega_2) = c_2 \tag{2.7}$$

where c_1, c_2 are constants. Thus the design of the 2-D filter is reduced to the design of the transform $f_{\mathrm{OM}}(\omega_1, \omega_2)$.

We now proceed as follows: we will assume that we have a 1-D zero phase low-pass filter with cutoff frequency ω_0 and we will define the coefficients of (2.4) in such a way that the 2-D passband and stopband are mapped in the 1-D passband and stopband, respectively.

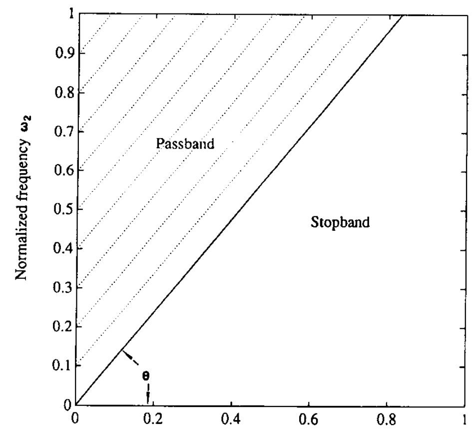

Fig. 1. Specifications of an ideal fan filter within the first quadrant.

As we said, in order to have full coverage of the 1-D band, $f_{\mathrm{OM}}(\omega_1, \omega_2)$ must satisfy (2.3). Finding constraints on the coefficients of the transform such that the equalities in (2.3) are satisfied is not an easy problem. In other words it is difficult to find the locations where the transform attains its global maximum and minimum values.

A solution to the above problem, which was introduced by Mersereau in [14], [15], [28], is to disregard (2.3) and calculate the coefficients using some optimization technique. Then the transform is scaled in order the transform in (2.4) to have values in the range $[-1,1]$. The new transform is used for the design of desired filter. Later Nguyen and Swamy in [12], [13] described simple formulas for scaling and they finally gave a scaling free McClellan transform. This transform permits the design of circular and elliptical shaped filters without the need of scaling.

In the next section we will define conditions that the McClellan transform must satisfy in order not to need scaling. These conditions are applicable only for the case of fan filters design.

III. The McClellan Transform and the Fan Filters

Let ω_0, be the cut-off frequency of the 1-D prototype zero phase low-pass filter. Let also θ, be the angle given by the 2-D fan filter specifications, (as shown in Fig. 1). Since we have quadrantal symmetry, we will limit ourselves only to the first quadrant. Notice that the $(0,\pi), (\pi,0)$ points are in the passband and stopband, respectively. Thus we require the original McClellan transform (2.4) to satisfy the following condition:

$$-1 = f_{\mathrm{OM}}(\pi, 0) \leqslant f_{\mathrm{OM}}(\omega_1, \omega_2) \leqslant f_{\mathrm{OM}}(0, \pi) = 1. \tag{3.1}$$

In other words, we require the global maximum and global minimum of the transform to be attained at the $(0,\pi)$ and $(\pi,0)$ points, respectively. This means that we are mapping the $(0,\pi)$ point of the 2-D frequency plane to the 1-D frequency origin and the $(\pi,0)$ point of the 2-D frequency plane to the π point of the 1-D frequency. In order to have $f_{OM}(\pi,0)=-1$ and $f_{OM}(0,\pi)=1$ it is easy to verify that the coefficients of (2.4) must satisfy

$$\begin{aligned} t_{11} &= t_{00} \\ t_{10} &= 1+t_{01}. \end{aligned} \tag{3.2}$$

Under (3.2), (2.4) is written as follows:

$$F_{OM}(\omega_1,\omega_2) = t_{11}(1+\cos(\omega_1)\cos(\omega_2)) + (t_{01}+1)\cos(\omega_1) + t_{01}\cos(\omega_2). \tag{3.3}$$

Notice that $F_{OM}(\omega_1,\omega_2)$, depends only on two parameters while the $f_{OM}(\omega_1,\omega_2)$ had four parameters.

We would like to find conditions on t_{01}, t_{11} such that our transform $F_{OM}(\omega_1,\omega_2)$ satisfies condition (3.1), for every pair (ω_1,ω_2) in the 2-D frequency plane; namely

$$-1 \leqslant F_{OM}(\omega_1,\omega_2) \leqslant 1. \tag{3.4}$$

Notice that if (3.4) holds we assure that the whole 2-D frequency plane is mapped onto the whole 1-D band and thus our transform does not need scaling as was the case in [10], [14], [28]. A necessary and sufficient condition for the validity of (3.4), is given in the following theorem.

Theorem: The McClellan transform $F_{OM}(\omega_1,\omega_2)$ is absolutely bounded by unity for every ω_1,ω_2 in the 2-D frequency plane, i.e., $|F_{OM}(\omega_1,\omega_2)| \leqslant 1$, $\forall\ \omega_1,\omega_2 \in [0,\pi]\times[0,\pi]$, if and only if the following constraint on the coefficients holds:

$$|t_{11}| \leqslant \min\{(1+t_{01}), -t_{01}\} \tag{3.5}$$

Proof: Our goal is to satisfy (3.4). In other words we want to have for every pair $(\omega_1,\omega_2)\in[0,\pi]\times[0,\pi]$

$$\begin{aligned} -1 \leqslant\, & t_{11}(1+\cos(\omega_1)\cos(\omega_2)) \\ & + (t_{01}+1)\cos(\omega_1) + t_{01}\cos(\omega_2) \\ \leqslant\, & 1. \end{aligned} \tag{3.6}$$

Consider first the left-hand side (LHS) inequality of (3.6). After some mathematical manipulations this inequality is equivalent to

$$-\frac{1+\cos(\omega_1)}{1+\cos(\omega_1)\cos(\omega_2)}[1+t_{01}(1+\cos(\omega_2))] \leqslant t_{11}-t_{01}. \tag{3.7}$$

In order (3.7) to be true for every $(\omega_1,\omega_2)\in[0,\pi]\times[0,\pi]$, it is necessary and sufficient to be true for the maximum value of the LHS expression. The maximum value $L_{\max}$, of the LHS part, depends on the value of the coefficient t_{01} as follows:

$$L_{\max} = \begin{cases} -(1+2t_{01}), & \text{for } t_{01} \leqslant -0.5 \\ 0, & \text{for } t_{01} > -0.5. \end{cases} \tag{3.8}$$

Thus, using (3.8), (3.7) is equivalent to

$$t_{11} \geqslant \begin{cases} -(1+t_{01}), & \text{for } t_{01} \leqslant -0.5 \\ t_{01}, & \text{for } t_{01} > -0.5. \end{cases} \tag{3.9}$$

Consider now the right-hand side (RHS) inequality of (3.6), this is equivalent to

$$\frac{1-\cos(\omega_1)}{1+\cos(\omega_1)\cos(\omega_2)}[1+t_{01}(1-\cos(\omega_2))] \geqslant t_{11}+t_{01}. \tag{3.10}$$

In a similar way,

$$t_{11} \leqslant \begin{cases} (1+t_{01}), & \text{for } t_{01} \leqslant -0.5 \\ -t_{01}, & \text{for } t_{01} > -0.5. \end{cases} \tag{3.11}$$

Finally, combining the necessary and sufficient constraints (3.9) and (3.11) we get

$$|t_{11}| \leqslant \begin{cases} (1+t_{01}), & \text{for } t_{01} \leqslant -0.5 \\ -t_{01}, & \text{for } t_{01} > -0.5 \end{cases} \tag{3.12}$$

but (3.12) is equivalent to (3.5) and this concludes the proof of our theorem.

Comment: Notice that (3.5) can be true only for $-1 \leqslant t_{01} \leqslant 0$. Also since $F_{OM}(0,\pi)=1$ and $F_{OM}(\pi,0)=-1$, if (3.5) holds then this means that $F_{OM}(\omega_1,\omega_2)$ has a global maximum at $(0,\pi)$ and a global minimum at $(\pi,0)$.

IV. Designing Zero Phase Fan Filters Via the McClellan Transform

In this section we will define the coefficients of the McClellan transform in order to design 2-D zero phase FIR fan filters. The ideal specifications of a 2-D fan filter with quadrantal symmetry is shown in Fig. 1. As we can note, there is a line, (let us call it a cut-off line), which separates the passband from the stopband region. Let θ, be the angle of this line with the ω_1-axis. The angle θ can take values in $(0,\pi/2)$, in order to produce fan filters of arbitrary inclination.

Our goal is the following: to a given $\theta\in(0,\pi/2)$, select the values of the coefficients t_{01} and t_{11} of the transform (3.3) and the cutoff frequency ω_0, such that the resulting iso-potential corresponding to ω_0, approximates the cut-off line. The equation of the cut-off line is given by

$$\omega_2 = \kappa\omega_1 \tag{4.1}$$

where

$$\kappa = \tan(\theta). \tag{4.2}$$

The equation for the iso-potentials can be derived by solving (3.3) for ω_2 as a function of ω_1, and is given by

$$\omega_2 = g(\omega_1) = \arccos\left(\frac{\cos(\omega)-(1+t_{01})\cos(\omega_1)-t_{11}}{t_{01}+t_{11}\cos(\omega_1)}\right). \tag{4.3}$$

An obvious way to formulate the design problem is to require (4.3) to approximate (4.1) for $\omega=\omega_0$. If we define proximity in the mean square or min–max sense, then this leads to a nonlinear problem, the solution of which requires a large computational effort. In the following we propose a different formulation that leads to a linear problem. Define a deviation function $D(\omega_1,\omega_2,\omega)$ with the relation

$$D(\omega_1,\omega_2,\omega) = F_{OM}(\omega_1,\omega_2) - \cos(\omega). \tag{4.4}$$

We know that a function $\omega_2 = g(\omega_1)$ is an iso-potential for some $\omega\in[0,\pi]$, if $D(\omega_1,\omega_2,\omega)=0$. Since we want $\omega_2=\kappa\omega_1$ to be an iso-potential for $\omega=\omega_0$ we must have $D(\omega_1,\kappa\omega_1,\omega_0)=0$. Unfortunately, this requirement, in the general case, is not met by any values of t_{01}, t_{11}, and ω_0. Thus we must define t_{01}, t_{11}, and ω_0, in order to minimize

$D(\omega_1, \kappa\omega_1, \omega_0)$ in some sense. To reduce the number of parameters of the above minimization problem, i.e., to simplify the computation of the unknowns, we define the cut-off frequency ω_0 of the prototype 1-D filter as a function of the angle θ. To this end let us see the form of the iso-potentials. Consider ω_1, ω_2 to have small values and let us make the approximation $\cos(v) \approx 1 - v^2/2$, then (3.3), after keeping only second-order terms, is equivalent to

$$\frac{\omega_2^2}{a^2} - \frac{\omega_1^2}{b^2} = \cos(\omega) - (2p+1) \tag{4.5}$$

where

$$a^2 = (-2/p) \tag{4.6}$$

$$b^2 = 2/(p+1) \tag{4.7}$$

and

$$p = t_{01} + t_{11}. \tag{4.8}$$

From (4.5) we can see that the iso-potentials, for small ω_1, ω_2, are hyperbolas. Let us now concentrate on the case $\omega = \omega_0$. For this case, as we said, we want the corresponding iso-potential to be close to (4.1). We can thus require our iso-potential to satisfy (4.1) at least for small ω_1, ω_2. In other words we would like (4.5) to be equivalent to (4.1). This is the case when

$$\cos(\omega_0) = 2p + 1 \tag{4.9}$$

$$a/b = \kappa. \tag{4.10}$$

Using (4.6)–(4.10) we conclude that

$$p = -\frac{\cos(2\theta) + 1}{2} = t_{01} + t_{11} \tag{4.11}$$

and

$$\cos(\omega_0) = -\cos(2\theta) \tag{4.12}$$

or

$$\omega_0 = \pi - 2\theta. \tag{4.13}$$

Relation (4.13) defines the cut-off frequency ω_0 as a function of the angle θ. Also (4.11) constitutes a constraint on the sum of the coefficients t_{01} and t_{11}. Thus we have to define only one unknown and we select t_{11}.

We can now reformulate the minimization problem as follows:

$$\hat{D}(\omega_1, \omega_2) = A(\omega_1, \omega_2) t_{11} + B(\omega_1, \omega_2) \tag{4.14}$$

where

$$A(\omega_1, \omega_2) = (1 - \cos(\omega_1))(1 - \cos(\omega_2)) \tag{4.15a}$$

$$B(\omega_1, \omega_2) = -[(p+1)(1 - \cos(\omega_1)) + p(1 - \cos(\omega_2))]. \tag{4.15b}$$

Define

$$S(t_{11}) = \int_0^u \hat{D}^2(\omega_1, \kappa\omega_1)\, d\omega_1 \tag{4.16}$$

where $u = \pi$ for $\theta \in (0, \pi/4]$ and $u = \pi/\kappa$ for $\theta \in (\pi/4, \pi/2)$. We will limit ourselves to the case $\theta \in (0, \pi/4]$. For angles $\theta \in (\pi/4, \pi/2)$ we can show that the coefficients t_{01} and t_{11} are given by

$$t_{11} = -\hat{t}_{11}$$
$$t_{01} = -(1 + \hat{t}_{01}) \tag{4.17}$$

where $\hat{t}_{01}, \hat{t}_{11}$ are the coefficients computed for $\hat{\theta} = \pi/2 - \theta$. Now we want to minimize $S(t_{11})$ defined in (4.16). Since the minimization is with respect to only one parameter, we can find the analytic solution

$$t_{11} = -\frac{\int_0^\pi A(\omega_1, \kappa\omega_1) B(\omega_1, \kappa\omega_1)\, d\omega_1}{\int_0^\pi A^2(\omega_1, \kappa\omega_1)\, d\omega_1}. \tag{4.18}$$

After the analytic evaluation of the integrals, we obtain

$$t_{11} = \frac{N_1 + N_2 \sin(\kappa\pi) + N_3 \sin(2\kappa\pi)}{D_1 + D_2 \sin(\kappa\pi) + D_3 \sin(2\kappa\pi)} \tag{4.19}$$

where

$$N_1 = \frac{3\pi(2p+1)}{2}$$
$$N_2 = \frac{2(2p+1)\kappa}{1-\kappa^2} - \frac{3p+1}{\kappa} - \frac{(p+1)(2-\kappa^2)}{\kappa(4-\kappa^2)} \tag{4.20a}$$
$$N_3 = \frac{p}{4\kappa} - \frac{\kappa p}{1 - 4\kappa^2}$$

and

$$D_1 = \frac{9\pi}{4}$$
$$D_2 = -\frac{2}{\kappa}\left(1 + \frac{2-\kappa^2}{4-\kappa^2} - \frac{2\kappa^2}{1-\kappa^2}\right)$$
$$D_3 = \frac{1}{4\kappa}\left(1 + \frac{1-2\kappa^2}{2(1-\kappa^2)} - \frac{8\kappa^2}{1-4\kappa^2}\right). \tag{4.20b}$$

Notice that for $\theta = 45°$, we have $B(\omega_1, \kappa\omega_1) = 0$ for all ω_1 and thus for this case $t_{11} = 0$. This gives a mean squared error equal to zero which means that our solution is exact. This case was treated in [8]. In Table I we present the values of t_{01}, t_{11} for different values of θ. In Figs. 2, 3, and 4 we have plotted the iso-potentials for $\theta = 20$, 30, and 45 degrees, respectively. Notice that the form of the iso-potentials is very close to a hyperbola not only for small values of ω_1, ω_2. From the table we conclude that the proposed design method gives results with small MSE.

TABLE I
THE VALUES OF MCCLELLAN TRANSFORM COEFFICIENTS AND THE CORRESPONDING MSE AND RELATIVE ABSOLUTE DEVIATION E, FOR DIFFERENT VALUES OF THE ANGLE θ

θ	t_{01}	t_{11}	MSE	E (%)
5	-.737691	-.254713	6.14×10^{-7}	1.8
10	-.725172	-.244673	9.43×10^{-6}	1.8
15	-.704518	-.228494	4.44×10^{-5}	1.8
20	-.676847	-.206174	1.25×10^{-4}	1.8
25	-.643468	-.177925	2.57×10^{-4}	1.8
30	-.606136	-.143863	4.07×10^{-4}	1.7
35	-.567235	-.103774	4.72×10^{-4}	1.5
40	-.530125	-.056699	2.99×10^{-4}	1.2
45	-.500000	0	7.97×10^{-13}	0

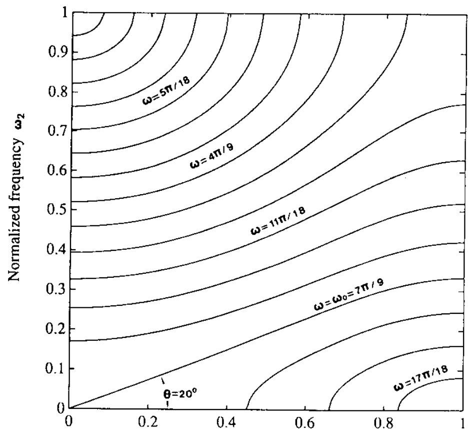

Fig. 2. Iso-potentials obtained from the computed coefficients of the McClellan transform. $\theta = 20°$.

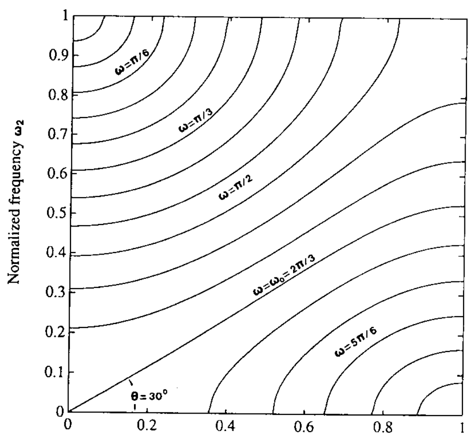

Fig. 3. Iso-potentials obtained from the computed coefficients of the McClellan transform. $\theta = 30°$.

Unfortunately this error does not give a measure of how close our approximate cut-off iso-potential to the ideal cutoff line is. Thus we also compute the relative absolute deviation of our iso-potential with the ideal one, as follows:

$$E = \frac{\int_0^{\pi} |\kappa\omega_1 - g(\omega_1)|\, d\omega_1}{\int_0^{\pi} \kappa\omega_1\, d\omega_1} \times 100 \tag{4.21}$$

where the function $\kappa\omega_1$ describes the ideal cut-off line and $g(\omega_1)$ describes our cut-off isopotential given by (4.3). The values of E are also given in Table I. Notice that the relative absolute deviation is very small.

In order our proposed design method to be complete, we must prove that the values of the coefficients t_{01} and t_{11} resulting from our design, satisfy the constraint (3.5) in the theorem of Section III. With the next lemma we prove that this is indeed the case.

Lemma: The coefficients t_{01} and t_{11} defined by (4.11) and (4.18) satisfy the constraint (3.5).

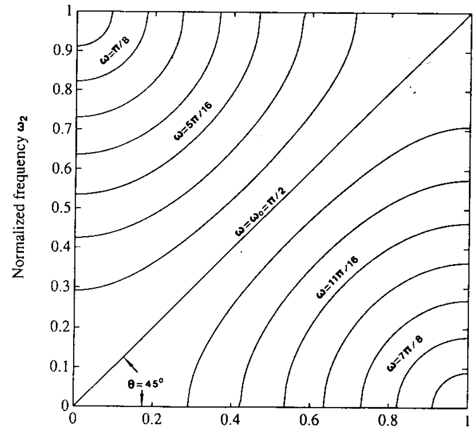

Fig. 4. Iso-potentials obtained from the computed coefficients of the McClellan transform. $\theta = 45°$.

Proof: Consider the following cases.

1) *Case* $\theta \in (0, \pi/4]$: From (4.11) and using the relation $-1 \leqslant \cos(2\theta) \leqslant 1$, we conclude that the value of the coefficient t_{11} which results from our design method, meet the constraint

$$-(1+t_{01}) \leqslant t_{11} \leqslant -t_{01}. \tag{4.22}$$

Thus using (4.22), in order t_{11} to subject to the constraint (3.5) it is sufficient to prove that t_{11} also satisfies the following constraint:

$$t_{01} \leqslant t_{11} \leqslant (1+t_{01}) \tag{4.23}$$

because then we show that $|t_{11}| \leqslant t_{01}$ and $|t_{11}| \leqslant (1+t_{01})$. Adding t_{11} to all members of (4.23), using (4.12) and the relation $\cos(2\theta)+1 = 2/(1+\tan^2(\theta))$, (4.23) can be written as

$$\frac{-1}{2(1+\kappa^2)} \leqslant t_{11} \leqslant \frac{\kappa^2}{2(1+\kappa^2)}. \tag{4.24}$$

Consider first the LHS inequality of (4.24). Substitution of t_{11} from (4.18) into (4.24), yields

$$\int_0^{\pi} A(\omega_1, \kappa\omega_1)\Big[\kappa^2(1-\cos(\omega_1)) - (1-\cos(\kappa\omega_1)) + \frac{(1-\cos(\omega_1))(1-\cos(\kappa\omega_1))}{2}\Big]\, d\omega_1 \geqslant 0. \tag{4.25}$$

Since from (4.15) $A(\omega_1, \kappa\omega_1)$ is non-negative, we can show that (4.25) is true if we show that the quantity contained in the brackets is non-negative. Because we are in the first quadrant and $0 \leqslant \kappa \leqslant 1$, using the relation $1-\cos(2a) = 2\sin^2(a)$, the quantity in the brackets is non-negative if

$$\frac{\tan\left(\frac{\omega_1}{2}\right)}{\frac{\omega_1}{2}} \geqslant \frac{\sin\left(\frac{\kappa\omega_1}{2}\right)}{\frac{\kappa\omega_1}{2}}. \tag{4.26}$$

The function $\tan(x)/x$ is a monotone increasing function of x in the range $[0, \pi/2]$, therefore, we have

$$\frac{\tan\left(\frac{\omega_1}{2}\right)}{\frac{\omega_1}{2}} \geqslant \frac{\tan\left(\frac{\kappa\omega_1}{2}\right)}{\frac{\kappa\omega_1}{2}} \geqslant \frac{\sin\left(\frac{\kappa\omega_1}{2}\right)}{\frac{\kappa\omega_1}{2}} \tag{4.27}$$

Thus the LHS inequality of (4.24) is true. Considering now the RHS inequality of (4.24), in a similar way we can show that it is true if

$$\frac{\tan\left(\frac{\kappa\omega_1}{2}\right)}{\frac{\kappa\omega_1}{2}} \geqslant \frac{\sin\left(\frac{\omega_1}{2}\right)}{\frac{\omega_1}{2}}. \tag{4.28}$$

Since the function $\sin(x)/x$ is a monotone decreasing function of x in the range $[0, \pi/2]$, we have

$$\frac{\tan\left(\frac{\kappa\omega_1}{2}\right)}{\frac{\kappa\omega_1}{2}} \geqslant \frac{\sin\left(\frac{\kappa\omega_1}{2}\right)}{\frac{\kappa\omega_1}{2}} \geqslant \frac{\sin\left(\frac{\omega_1}{2}\right)}{\frac{\omega_1}{2}} \tag{4.29}$$

and this concludes the proof of (4.24).

ii) *Case* $\theta \in (\pi/4, \pi/2)$*:* Starting from (3.5) and (4.17), it is straightforward to prove that the coefficients t_{01} and t_{11}, meet the constraint (3.5) and this concludes the proof of the lemma.

With the above lemma we have proved that our design does not need scaling and also that we use the whole 1-D band.

V. Design Procedure

In this section we will apply our design method of Section IV, to the design of 2-D zero phase FIR fan filters with quadrantal symmetry. Then we will extend our method to the design of fan filters of more general shape without quadrantal symmetry.

To design a 2-D zero phase FIR fan filter with quadrantal symmetry such as shown in Fig. 1 with our method, we must proceed according to the following steps.

Step 1: From the angle θ, i.e., the inclination of the desired fan filter, using (4.13) we define the desired 1-D cutoff frequency ω_0.

Step 2: Using some 1-D filter design method, we design a zero phase low-pass FIR filter with cutoff frequency ω_0.

Step 3: Using (4.11) and (4.18) we compute the appropriate values of the McClellan transform coefficients.

Step 4: We write the frequency response of the 1-D designed filter in the form $\hat{G}(e^{j\omega}) = \Sigma_n g_n T_n[\cos(\omega)]$, where $T_n[x]$ is the nth-order Chebychev polynomial. We then substitute the $\cos(\omega)$ in the frequency response of the prototype filter, with the McClellan transform (3.3) to produce the desired 2-D zero phase FIR fan filter.

General Shape 2-D Fan Filters

A fan filter with more general shape is shown in Fig. 5. For uses of this type of filters see [18]. To design such a filter, we follow the above-mentioned procedure twice with $\theta = \theta_1$ and $\theta = \theta_2$, respectively. If we define a filter with frequency response equal to the difference of the two

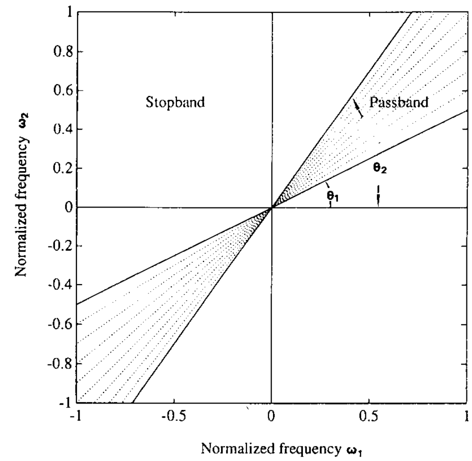

Fig. 5. Specifications of an ideal general shape fan filter.

frequency responses, that is $\hat{H}(e^{j\omega_1}, e^{j\omega_2}) = \hat{H}_1(e^{j\omega_1}, e^{j\omega_2}) - \hat{H}_2(e^{j\omega_1}, e^{j\omega_2})$, where the frequency response $\hat{H}_1(e^{j\omega_1}, e^{j\omega_2})$ corresponds to the angle θ_1 and the frequency response $\hat{H}_2(e^{j\omega_1}, e^{j\omega_2})$ corresponds to the angle θ_2, then the resulting frequency response is quadrantally symmetric with the first and the third quadrants being the same as the first and the third quadrants of Fig. 5. We thus need to reject the second and the fourth quadrants. This can be achieved by cascading a "quadrant fan filter" [14], [24], [28]. This filter can be designed using a transform method. For example we can design such a filter using a 1-D zero phase low-pass FIR filter with cut-off frequency $\omega_0 = \pi/2$ and the McClellan transform $f_{GM}(\omega_1, \omega_2) = \sin(\omega_1)\sin(\omega_2)$, [14], [28]. Notice that the "quadrant fan filter" does not depend on the specific fan specifications and thus can be designed only once.

VI. Conclusion

In this paper, a new design method was presented that computes the McClellan transform coefficients needed for the design of 2-D zero phase FIR fan filters with quadrantal symmetry. The coefficient values were computed using, e.g., MSE, as an optimality criterion. The resulting cut-off iso-potential of our design method were shown to have a very small relative absolute deviation from the ideal one. Extension of our method to general shape fan filters were also given.

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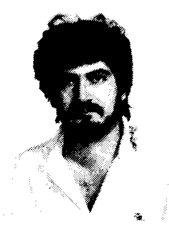

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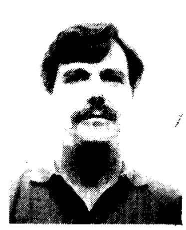

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